Ricci-flat deformations of orbifolds and asymptotically locally Euclidean manifolds

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the acknowledgements and specified in the text is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the acknowledgements and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as declared in the acknowledgements and specified in the text.

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Abstract

In this thesis we study Ricci-flat deformations of Ricci-flat Kähler metrics on compact orbifolds and asymptotically locally Euclidean(ALE) manifolds. In both cases we also study the moduli space of Ricci-flat structures. For this purpose, it is convenient to assume that the initial Ricci-flat metrics are Kähler. Our work extends results by Koiso about Einstein-deformations of Kähler-Einstein metrics on compact manifolds.

Orbifolds differ from manifolds by being locally modelled on a quotient of Euclidean space by the action of a finite group Γ. We adapt a slice construction by Ebin and the Calabi conjecture to orbifolds and show that for compact complex orbifolds with vanishing orbifold first Chern class and all infinitesimal complex deformations integrable, Ricci-flat deformations of Ricci-flat Kähler metric are Kähler, possibly with respect to a perturbed complex structure. We also show that the moduli space of Ricci-flat structures is, up to the action of a finite group, a finite dimensional manifold and we express its dimension in terms of the dimension of certain Dolbeault and sheaf cohomology groups. The strategy is to lift the problem locally to a Γ-invariant problem on a manifold.

ALE manifolds are non-compact manifolds with one end, for which the metric at infinity approximates a flat metric. We study ALE Ricci-flat Kähler manifolds that arise as the complement of a divisor D in a compact Kähler manifold $\bar{X}$ and use the deformation theory by Kawamata for the pair $(\bar{X}, D)$. By working with suitably chosen weighted Sobolev and Hölder spaces we recover the relevant elliptic theory for the linearisation of the Ricci operator and the linearisation of the complex Monge-Ampère equation. We prove that integrability of infinitesimal deformations of the pair $(\bar{X}, D)$ implies that ALE Ricci-flat deformations of ALE Ricci-flat Kähler metrics are Kähler, possibly with respect to a perturbed complex structure. We also show that the moduli space of ALE Ricci-flat structures is, up to the action of a finite group, a finite dimensional manifold and we express its dimension in terms of the dimension of certain Dolbeault and sheaf cohomology groups.

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Contents

Chapter 1

Introduction

The starting point for this thesis is the basic observation that the set of Riemannian metrics $\mathcal{M}$ on a smooth manifold M is vast and that there a priori is no preferred choice of metric. The set of Riemannian structures on M is the quotient $\widetilde{\mathcal{M}} = \mathcal{M}/\mathcal{D}$, where $\mathcal{D}$ denotes the group of diffeomorphism of M. In [BES87, Chapter 0] Besse asked, for a compact smooth manifold M,

> Are there any best (or nicest, or distinguished) Riemannian structures on M?

For manifolds of dimension two the uniformization theorem provides an answer. Every connected two-dimensional manifold admits a complete metric with constant Gaussian curvature. In higher dimensions the concept of curvature is more complicated, and so is any potential answer involving a curvature condition. A single 'distinguished' Riemannian structure seems out of reach on higher dimensional manifolds in general, but one could hope that a suitable curvature condition could provide a collection of 'best' Riemannian structures which is significantly smaller than $\widetilde{\mathcal{M}}$. The three main types of curvature are sectional curvature, Ricci curvature and scalar curvature. All manifolds admit constant scalar curvature metrics, but they come in such abundances that the space of constant scalar curvature metrics is infinite dimensional in general([BES87, section 4.F]). A constant scalar curvature condition therefore seems too weak. On the other had, assuming constant sectional curvature is too restrictive, as many manifolds do not admit any constant sectional curvature metrics. This leaves us with a condition on the Ricci-curvature. The Einstein condition is a condition on the Ricci curvature. If we normalise volume, Einstein metrics are critical points of the total scalar curvature functional, which at least partially merits a status as a distinguished metric. Also, according to [BES87, remark 0.21] no compact manifolds of dimension ≥ 5 are known not to admit an Einstein metric.

We take a closer look at the set of Einstein structures. Denote by $\mathcal{M}_1$ the metrics with volume 1. For a connected compact Riemannian manifold (M, g) Ebin has shown that there almost exists a slice $S \subset \mathcal{M}_1$ for the action of $\mathcal{D}$, i.e. the space $\widetilde{\mathcal{M}_1}$ is locally homeomorphic to $S/\text{Iso}(M, g)$, where $\text{Iso}(M, g)$ denotes the group of isometries. If we denote by $\mathcal{R}$ the subspace of Einstein metrics in $\mathcal{M}_1$, then the quotient space $\widetilde{\mathcal{R}} = \mathcal{R}/\mathcal{D}$ is called the moduli space of Einstein structures. Koiso carried out a detailed study of this moduli space in [KOI83]. He showed that for a slice through an Einstein metric g there is a finite dimensional manifold inside the slice containing the nearby Einstein metrics of the slice as a real analytic subset. If we furthermore assume that the compact Einstein manifold (M, g) is Kähler, has non-positive first Chern class and that all infinitesimal complex deformations are integrable, then Koiso showed that all infinitesimal Einstein deformations will in fact integrate into curves of Einstein metrics. This implies that all Einstein deformations of a Kähler-Einstein metric are Kähler, possibly with respect to a perturbed complex structure. When the first Chern class is non-positive then the isometry group $\text{Iso}(M, g)$ acts as a finite group on the space of Einstein metrics in the slice S. It follows that the local model for the moduli space of Einstein structures is, up to an action of a finite group, a finite dimensional manifold. Koiso also found an expression for its dimension in terms of cohomology groups. Einstein structures therefore seem like a decent candidate for a 'distinguished' Riemannian structure on a compact manifold.

Despite the innocently looking condition $\text{Ric}(g) = \lambda g$, Einstein metrics are notoriously hard to construct, [BES87, 0.23]. On Kähler manifolds we have a powerful tool though. The Calabi conjecture [CAL54] constructs Ricci-flat Kähler metrics on compact Kähler manifolds with vanishing first Chern class. The combination of the Calabi conjecture and the results by Koiso give us a decent understanding of Einstein structures which is part of the reason for promoting them as a candidate for a distinguished class of structures. The focus of this thesis is to explore if this understanding holds on more general objects than compact manifolds. A lot of work has already been done on proving the Calabi conjecture on more general spaces, so our main objective is to generalize the results by Koiso. The sign of the first Chern class plays a role in the understanding of Einstein metrics in general and in particular also for Koiso's results. When the sign is negative or zero the problem can be solved with the above assumptions. When the sign is positive one needs the extra assumption that no holomorphic vector fields exist. According to [BES87, 12.101] no such manifolds are known.

In this thesis we explore two ways of relaxing the hypothesis of Koiso's results. In Chapter 3 we show that similar results can be obtained when the compact manifold is replaced by a compact orbifold. In Chapter 5 we show that a version of Koiso's results can be proved for a class of non-compact manifolds known as asymptotically locally Euclidean manifolds.

An orbifold generalizes the concept of a manifold by admitting a more complicated local model. An orbifold is locally homeomorphic to a quotient of Euclidean space by a finite group. The action might have isotropy, which lead to the presence of quotient singularities. At such singular points the structure of the orbifold differs from that of a manifold. Orbifolds can be considered natural in the sense that they arise as the result of some natural operations in differential geometry. For instance, Koiso's results show that the moduli space of Einstein structures on a compact manifold is an instance of an orbifold. It seems natural to ask if the understanding of Einstein metrics we have on manifolds also holds for orbifolds. In this thesis we provide an affirmative answer to this question. We show that for a compact complex orbifold with vanishing first Chern class, integrability of infinitesimal complex deformations implies that Ricci-flat deformations of Ricci-flat Kähler metrics are Kähler, possibly with respect to a perturbed complex structure. We also show that the moduli space of Ricci-flat structures in a neighbourhood of a Ricci-flat Kähler structure is, up to the action of a finite group, a finite dimensional manifold and we find an expression for the dimension of it. For completeness we also provide a proof of the Calabi conjecture on orbifolds. Locally an orbifold is a quotient of a manifold by a finite group Γ. Our strategy is locally to lift the problem on an orbifold to a Γ-invariant problem on a manifold.

An asymptotically locally Euclidean (ALE) manifold is a non-compact manifold together with a metric which at infinity approximates a flat metric. Proving a version of Koiso's results for non-compact manifolds provide a number of challenges. For instance, it is used that elliptic operators are Fredholm, which need not be true on non-compact manifolds. For non-compact manifolds there is also no Kodaira-Spencer stability result saying that a smooth family of complex deformations through a complex structure with a Kähler metrics admits a smooth family of compatible Kähler metrics. To remedy the problems related to the elliptic operators we work with appropriately chosen weighted Hölder and Sobolev spaces of sections. To tackle the deformation theory of complex structures we consider ALE manifolds that arise as the complement of a smooth divisor in a compact Kähler manifold and use the deformation theory by Kawamata for such pairs. We consider an ALE Ricci-flat Kähler manifold $X = \bar{X} \backslash D$ where $\bar{X}$ is a compact Kähler manifold and D is a smooth divisor satisfying $K_{\bar{X}} = -\beta L_D$ for some $\beta \geq 1$. We show that if all infinitesimal deformations of the pair $(\bar{X}, D)$ are integrable, then ALE Ricci-flat deformations of g are Kähler, possibly with respect to a perturbed complex structure. We also show that the moduli space of ALE Ricci-flat structures in a neighbourhood of a Ricci-flat Kähler structure is, up to the action of a finite group, a finite dimensional manifold. Joyce proved in [JOY00, Chapter 8] that the Calabi conjecture also holds for ALE metrics.

The basic understanding of Einstein metrics regarding the existence of Einstein metrics on Kähler manifolds, their deformation theory and the finiteness of the dimension of their moduli space we get from the combination of the Calabi conjecture and the results by Koiso therefore carry over to orbifolds and ALE manifolds.

It should be mentioned that our inspiration for studying Ricci-flat Kähler metrics on compact orbifolds and ALE manifolds is not the only one. Another reason for studying Ricci-flat Kähler metrics on compact orbifolds and ALE manifolds is the role they play in the construction of compact manifolds with holonomy G_2 and $Spin(7)$. This will not be treated in this thesis, but it is a vast field of active research. See for instance [JOY00, Chapter 11-15] and [KOV-NOR10].

This thesis is divided into five chapters. In Chapter 2 we present some background material. In Chapter 3 we generalize Koiso's results to compact orbifolds. In Chapter 4 we apply the orbifold version of Koiso's results to compute the dimension of the moduli space of Ricci-flat structures for some orbifold K3 surfaces. In Chapter 5 we prove a version of Koiso's results for ALE manifolds and in Chapter 6 we discuss examples of ALE manifolds.

Chapter 2

Preliminaries

In this chapter we give some background material and fix basic notation. The material is standard and is mostly taken from [LAN62], [BES87], [GRI-HAR94], [JOY00] and [HUY05].

All metrics in this thesis will be Riemannian and all finite dimensional manifolds will be of strictly positive dimension. We denote by (M, g) a smooth manifold with a metric g. If the manifold (M, g) is also complex, we denote it by (M, J) or (M, J, g). If it is furthermore Kähler, we use the symbol ω for the Kähler form of g with respect to J and we sometimes denote the manifold by (M, J, g, ω). We will usually use n for the real dimension of M and m for the complex dimension. Sometimes we write M^n to indicate that M is a smooth manifold of real dimension n. For a Riemannian manifold (M, g) we denote by $dV_g = d\mathrm{vol}_g$ the volume form of g and by $\mathrm{vol}(g)$ the total volume $\int_M dV_g$. We denote the tensor bundle on M by $T^{(r,s)}M$, where r is the covariant index and s the contravariant index, i.e. $T^{(0,1)}M = TM$ and $T^{(1,0)}M = T^*M$. Elements of $T^{(r,s)}M$ are called (r, s)-tensors or tensors of type (r, s). Covariant indices are raised and contravariant indices are lowered. In local coordinates $\{x^1, \ldots, x^n\}$ we write elements of TM as $X = \sum X^i \frac{\partial}{\partial x^i}$ and elements of T^*M as $\alpha = \sum \alpha_i dx^i$. Locally a metric g is given by $g_{ij} dx^i \otimes dx^j$. The component of the inverse matrix $(g_{ij})^{-1}$ we denote g^{ij}. We make use of the Einstein summation convention $\alpha_i X^i = \sum_{i=1}^n \alpha_i X^i$. Smooth sections of a tensor bundle $T^{(r,s)}M$ is denoted by $C^\infty(T^{(r,s)}M)$.

The musical isomorphisms, $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ are given by $X^\flat(Y) = g(X, Y)$ which in local coordinates reads $X^\flat = g_{ij} X^i dx^j$. The contravariant tensor $\alpha^\sharp$ is given by $\alpha^\sharp(\omega) = \omega(\alpha^\sharp)$, which in local coordinates reads $\alpha^\sharp = g^{ij} \alpha_j \frac{\partial}{\partial x^i}$.

The trace of a $(1,1)$-tensors ω is tr$(\sum_{i,j=1}^n \omega_j^i dx^i \otimes \frac{\partial}{\partial x^j}) = \sum_{i,j=1}^n \omega_j^i$. For a symmetric $(2,0)$-tensor h the trace with respect to the metric g is $\mathrm{tr}_g h = \mathrm{tr}\, h^\sharp = g^{ij}h_{ij}$. For complex tensors we denote by $\mathrm{tr}_\mathbb{C}$ the complex linear trace.

Denote by $\Omega^k(M)$ the space of *differential k-forms* on M and by $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the *exterior derivative*. The *differential*, *push-forward* or *linearization* of a smooth function $f : M \to N$ between smooth manifolds is the map $df_p : T_pM \to T_{f(p)}N$, where the map df_p in local coordinates is the Jacobi map $df_p = (\frac{\partial f_i}{\partial x_j})_{i,j=1,\dots,n}$. For $\alpha \in \Omega^1(N)$ the *pull-back* $f^*\alpha \in \Omega^1(M)$ is $f^*\alpha_p(X) = \alpha_{f(p)}(df_p(X))$. If f is a diffeomorphism and $\alpha \in \Omega^1(M)$ the *push-forward* $f_*\alpha \in \Omega^1(N)$ is $f_*\alpha_p(X) = \alpha_{f^{-1}(p)}((df_p^{-1}(X))$.

Using the metric g we can construct a pointwise inner product on $T^{(r,s)}M$ which we denote $(\cdot,\cdot)_p$. If M is compact there is an L^2-inner product on sections of $T^{(r,s)}M$ given by $(\eta,\rho)_{L^2} = \int_M (\eta_p,\rho_p)_p\, dV_g$. If there are more than one metric on M, then we denote the L^2-inner product with respect to g by $(\cdot,\cdot)_g$.

For general vector bundles $\pi : E \to M$ we denote by $C^\infty(E)$ the space of smooth sections. For a connection on E we denote by ∇ the corresponding *covariant derivative* $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$. We say that a tensor T is *parallel* if $\nabla T = 0$. For any metric g there exists a unique linear covariant derivative ∇ on TM satisfying $\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ and which is torsion free, i.e. $\nabla_X Y - \nabla_Y X - [X,Y] = 0$. It is the *Levi-Civita covariant derivative*.

For a complex manifold (M,J) we denote by $\mathbf{J}$ the multiplicative extension of J to $\wedge^* T_\mathbb{C} M$.We use Greek indices to express the following modified tensors. For the contravariant index a of the tensor $T^{a\dots}_{\dots}$ we define the tensors $T^{\alpha\dots}_{\dots} = \frac{1}{2}(T^{a\dots}_{\dots} - iJ^a_j T^{j\dots}_{\dots})$ and $T^{\bar\alpha\dots}_{\dots} = \frac{1}{2}(T^{a\dots}_{\dots} + iJ^a_j T^{j\dots}_{\dots})$. For the covariant index b of tensors $T^{\dots}_{b\dots}$ we define the tensors $T^{\dots}_{\beta\dots} = \frac{1}{2}(T^{\dots}_{b\dots} - iJ^j_b T^{\dots}_{j\dots})$ and $T^{\dots}_{\bar\beta\dots} = \frac{1}{2}(T^{\dots}_{b\dots} + iJ^j_b T^{\dots}_{j\dots})$. These operations are projections and they satisfy $T^{a\dots}_{\dots} = T^{\alpha\dots}_{\dots} + T^{\bar\alpha\dots}_{\dots}$ and $T^{\dots}_{b\dots} = T^{\dots}_{\beta\dots} + T^{\dots}_{\bar\beta\dots}$.

When we give definitions in the text we *emphasize them*. Throughout Chapter 2 we use the symbol M to denote a connected and oriented smooth manifold. It will often be compact, but not always. In Chapter 3 and 4 we use V for a smooth connected and oriented orbifold. It will often be compact, but not always. In Chapter 5 and 6 we use X for a smooth, connected and oriented non-compact manifold. It will often be ALE, but not always.

2.1 Tools from analysis

In this section we introduce various function spaces and fundamental results about them. The material is mostly borrowed from [JOY00, Chapter 1].

A *Banach space* is a vector space X together with a norm $||\cdot||_X$ for which the metric $d(x, y) = ||x - y||_X$ is complete. If the Banach norm is induced by an inner produce $(\cdot,\cdot)_X$, i.e. $||x|| = \sqrt{(x, x)_X}$, then $(X, (\cdot,\cdot)_X)$ is a *Hilbert space*. A Hausdorff topological vector space is a *Fréchet space* if its topology can be induces from a family of semi-norms $||\cdot||_k, k = 1, 2, \dots$ and its metric given by these semi-norms is complete.

We say that a topological space X is a *Hilbert-, Banach- or Fréchet manifold* if it admits an atlas of neighbourhoods homeomorphic to open sets in a Hilbert-, Banach- or Fréchet space respectively.

For a Riemannian manifold (M, g) denote by $L^1(M)$ the set of measurable function $f : M \to \mathbb{R}$ for which $\int_M f\, dV_g < \infty$. We say that a function $f : M \to \mathbb{R}$ is locally integrable if for all compact subsets $K \subset M$, $f \in L^1(K)$. We denote the space of *locally integrable functions* on M by $L^1_{loc}(M)$.

Definition 2.1.1. *For $p \geq 1$ define the Lebesgue space $L^p(M)$ to be the subset of $L^1_{loc}(M)$ for which the L^p-norm*

$$||f||_{L^p} = \left(\int_M |f|^p dV_g\right)^{\frac{1}{p}}$$

is finite.

Smooth functions on M with compact support are called *test functions*. The set of test functions is denoted by $C^\infty_c(M)$. A function $f \in L^1_{loc}(M)$ is said to be k-times *weakly differentiable* if for some $\tilde{f} \in L^1_{loc}(M)$, $\int_M f \nabla^\alpha \phi\, dV_g = (-1)^{|\alpha|} \int_M \tilde{f} \phi\, dV_g$ for all test functions $\phi \in C^\infty_c(M)$ and all multiindex $|\alpha| \leq k$.

Definition 2.1.2. *Define the* Sobolev space $L^p_k(M)$ *to be the set of those $f \in L^p(M)$ which are k-times weakly differentiable and for which $|\nabla^i f|_g \in L^p(M)$ for all $i = 1, \dots, k$. Equip $L^p_k(M)$ with the norm*

$$||f||_{L^p_k} = \left(\sum_{i=0}^{k} \int_M |\nabla^i f|^p\, dV_g\right)^{\frac{1}{p}}.$$

Denote by $C^k(M)$ the set of bounded continuous functions $f : M \to \mathbb{R}$ which have k times continuous bounded derivatives. Equip $C^k(M)$ with the norm

$$||f||_{C^k} = \sum_{i=0}^{k} \sup_M |\nabla^i f(x)|.$$

We say that a function $f : M \to \mathbb{R}$ is *Hölder continuous* with exponent $\alpha \in (0, 1)$ if

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty$$

where $d(x, y)$ is the distance from x to y on M using g. We can extend this notion of continuity to tensors over M in the following way. Let $\delta(g)$ be the *injectivity radius* of g. For $T \in T^{(r,s)}M$ and $x, y \in M$ with $d(x, y) < \delta(g)$ we can make sense of $|T(x) - T(y)|_g$ by choosing the unique geodesic joining x and y. Parallel translating along this geodesic we can identify $(T^{(r,s)}M)_x$ and $(T^{(r,s)}M)_y$ as vector spaces, so the subtraction and the norm makes sense.

Definition 2.1.3. *Define the* Hölder space $C^{k,\alpha}(M)$ *as the set of those* $f \in C^k(M)$ *for which* $\sup_M [\nabla^k f]_\alpha < \infty$. *Equip* $C^{k,\alpha}(M)$ *with the norm*

$$||f||_{C^{k,\alpha}} = ||f||_{C^k} + [\nabla^k f]_\alpha.$$

where for $k > 1$ *the supremum in* $[\nabla^k f]_\alpha$ *is taken inside the injectivity radius of the metric* g.

Lemma 2.1.4. *Let* (M, g) *be a Riemannian manifold. Let* $k \geq 0$ *and* $p > 1$ *be integers and* $\alpha \in (0, 1)$. *Equipping* $L^p(M), L^p_k(M), C^k(M)$ *and* $C^{k,\alpha}(M)$ *with the norms introduced above turns them into Banach spaces. The spaces* $L^2(M)$ *and* $L^2_k(M)$ *are also Hilbert spaces with respect to the inner products* $(f_1, f_2)_{L^2} = \int_M f_1 f_2 \, dV_g$ *and* $(f_1, f_2)_{L^2_k} = \sum_{j=0}^k \int_M (\nabla^j f_1)(\nabla^j f_2) \, dV_g$ *respectively.*

Lemma 2.1.4 also holds for the spaces of C^k, $C^{k,\alpha}$, L^p and L^p_k sections of vector bundles over M.

A normed vector space X is *continuously embedded* into a normed vector space Y if the inclusion map $\iota : X \to Y$ is continuous. A linear operator $A : (X, ||\cdot||_X) \to (Y, ||\cdot||_Y)$ is *compact* if for any sequence $\{x_n\}$ in X the sequence $\{Ax_n\}$ has a Cauchy subsequence.

We say that X is *compactly embedded* into Y if the inclusion map is compact. The next result is the Sobolev Embedding theorem (continuity) and Kondrachov's theorem (compactness) combined.

Theorem 2.1.5. *Let (M^n, g) be a compact Riemannian manifold. Let k, l be integers with $0 \leq l \leq k$ and p, q real numbers with $p, q \geq 1$ and $\alpha \in (0, 1)$. If*

$$\frac{1}{q} \leq \frac{1}{p} + \frac{k-l}{n}$$

Then $L^q_k(M)$ is continuously embedded into $L^p_l(M)$. If the inequality is strict then the embedding is also compact. If

$$\frac{1}{q} \leq \frac{k-l-\alpha}{n}$$

then $L^q_k(M)$ embeds continuously into $C^{l,\alpha}(M)$. If the inequality is strict, then the embedding is also compact. Also, for any $k \geq 0$ the embedding $C^{k,\alpha}(M) \to C^k(M)$ is compact.

Theorem 2.1.5 holds also for the space of sections of vector bundles over a compact manifold.

For vector bundles V and W over M a *linear differential operator* P of order k is a map that takes sections of V to sections of W and is of the form

$$Pu = A^{i_1 \dots i_k} \nabla_{i_1 \dots i_k} u + B^{i_1 \dots i_{k-1}} \nabla_{i_1 \dots i_{k-1}} u + \cdots + K^{i_1} \nabla_{i_1} u + Lu$$

where $\nabla_{i_1 \dots i_k} = \nabla_{i_1} \dots \nabla_{i_k}$ and $A, \dots, K$ are symmetric tensors and L is a smooth function. The *principal symbol* of P is $\sigma(P) : T^*M \times V \to W$ with $\sigma_\eta(P, x) = A^{i_1 \dots i_k}(x) \eta_{i_1} \dots \eta_{i_k} : V_x \to W_x$. It is a homogeneous polynomial of degree k in η. We say that P is *elliptic* if for all $x \in M$ and all $\eta \in T^*_x M$ with $\eta \neq 0$ we have $\sigma_\eta(P, x) \neq 0$. A non-linear differential operator is of the form $Pu = Q(x, u(x), \nabla u(x), \dots, \nabla^k u(x))$ for some continuous function Q.

Let P be a differential operator of order k and u a section of V with k derivatives. The *linearization* of P at u is

$$L_u P v = \frac{d}{dt}|_{t=0} P(u + tv) = \lim_{t \to 0} \frac{P(u + tv) - P(u)}{t}.$$

The linearized operator L_uP is again of order k. If P is a linear differential operator, then $L_uP = P$. We say that a non-linear differential operator is *elliptic* if its linearization is elliptic.

Theorem 2.1.6 (Elliptic regularity)**.** *Let (M^n, g) be a compact Riemannian manifold. Let V, W be vector bundles over M and let P be a smooth linear elliptic differential operator of order k. Assume that we have found $v \in L^1(V)$ and $w \in L^1(W)$ such that $P(v) = w$. Let p and l be integers with $p > 1$ and $l \geq 0$, and let $\alpha \in (0, 1)$.*

If $w \in C^\infty(W)$, then $v \in C^\infty(V)$.
If $w \in L^p_l(W)$, then $v \in L^p_{l+k}(V)$ and $||v||_{L^p_{l+k}} \leq C(||w||_{L^p_l} + ||v||_{L^1})$.
If $w \in C^{l,\alpha}(W)$, then $v \in C^{l+k,\alpha}(V)$ and $||v||_{C^{l+k,\alpha}} \leq C(|w||_{C^{l,\alpha}} + ||v||_{C^0})$.

In both cases $C > 0$ is some constant independent of v and w.

Theorem 2.1.7. *Let (M, g) be a compact manifold. Let V and W be vector bundles over M of the same dimension and let P be a smooth linear elliptic operator of order k from V to W. Let $l \geq 0$ and $p > 1$ be integers and $\alpha \in (0, 1)$. The operator P acts by $P : C^\infty(V) \to C^\infty(W)$, $P : L^p_{k+l}(V) \to L^p_l(W)$ and $P : C^{k+1,\alpha}(V) \to C^l(W)$. The kernel $\ker(P)$ is the same for each of the actions and it is a finite dimensional vector subspace of $C^\infty(V)$.*

Let V and W be vector bundles over a compact manifold M and let them be equipped with metrics. For a linear differential operator k from V to W there exists a unique linear operator P^* such that $(Pv, w)_W = (v, P^*w)_V$. This operator is called the *formal adjoint* operator of P.

Theorem 2.1.8 (Fredholm Alternative)**.** *Let (M, g) be a compact Riemannian manifold and let V, W be vector bundles over M equipped with metrics. Let P be a smooth linear elliptic operator from V to W of order k. Let $l \geq 0$ and $p > 1$ be integers and $\alpha \in (0, 1)$. Then the images of the maps*

$$P : L^p_{k+l}(V) \to L^p_l(W) \quad \textit{and} \quad P : C^{k+l,\alpha}(V) \to C^{l,\alpha}(W)$$

are closed vector subspaces of $L^p_l(W)$ and $C^{l,\alpha}(W)$ respectively.
For $w \in L^p_l(W)$ the equation $Pv = w$ admits a solution $v \in L^p_{k+l}(V)$ if and only if

$w \perp \ker(P^*)$. If $v \perp \ker(P)$ then v is unique.
For $w \in C^{l,\alpha}(W)$ the equation $Pv = w$ admits a solution $v \in C^{l+k,\alpha}(V)$ if and only if $w \perp \ker(P^*)$. If $v \perp \ker(P)$ then v is unique.

Theorem 2.1.9 (Inverse function theorem for Banach spaces)**.** *Let X, Y be Banach spaces and $U \subset X$ an open neighbourhood of $x \in X$. Suppose that the function $F : U \to Y$ is C^k for some $k \geq 1$, with $F(x) = y$, and that the first derivative $dF_x : X \to Y$ of F at x is an isomorphism between X and Y both as vector spaces and as topological spaces. Then there exist open neighbourhoods $U' \subset U$ of $x \in X$ and V' of $y \in Y$, such that $F : U' \to V'$ is a C^k-isomorphism.*

Theorem 2.1.10 (Implicit function theorem for Banach spaces)**.** *Let X, Y, Z be Banach spaces and let $U \subset X$ and $V \subset Y$ be open neighbourhoods of 0 in X and Y respectively. Assume that we have a C^k-function $F : U \times V \to Z$ with $F(0,0) = 0$. If $dF_{(0,0)}|_Y : Y \to Z$ is a linear homeomorphism, then there exists an open subset $U' \subset U$ and a unique C^k-map $G : U' \to Y$ such that $G(0) = 0$ and $F(x, G(x)) = 0$ for all $x \in U'$.*

Let U be a open subset of $\mathbb{R}^n$ and let V be an open subset of $\mathbb{R}^m$. A function $f : U \to V : x \mapsto (f_1(x), \ldots, f_m(x))$ is *real analytic* if around every point $p \in U$ there is a neighbourhood U' such that each component $f_j|_{U'}$ of $f|_{U'}$ admits a sequence of homogeneous polynomials P_i of degree i in n variables such that $f_j|_{U'}$ can be written as $f_j|_{U'}(x) = \sum_i P_i(x-p)$. A *real analytic* manifold M is a topological manifold for which all transition functions are real analytic. A *real analytic set* is the kernel of a real analytic map. The next result is a real analytic implicit function theorem. It is taken from [KOI83, Lemma 13.6].

Theorem 2.1.11. *Let V and W be Hilbert spaces and f a real analytic mapping from V to W defined on an open neighbourhood of the origin $0 \in V$. Assume that $f(0) = 0$ and that the image of the differential df_0 at 0 is closed in W. Then there is an open neighbourhood U of $0 \in V$ such that the set $f^{-1}(0) \cap U$ is a real analytic set in a real analytic submanifold Z of U whose tangent space $T_0 Z$ coincides with* $\ker(df_0)$.

2.2 Space of metrics

In this section we introduce the space of Riemannian metrics.

For a Riemannian manifold (M, g) denote by $\text{Sym}^2(T^*M)$ the tensor bundle of symmetric $(2, 0)$-tensors. Denote by $\mathcal{M}$ the set of smooth Riemannian metrics on M. In Section 2.1 the space $C^k(\text{Sym}^2(T^*M))$ was equipped with a Banach space topology. The space $C^\infty(\text{Sym}^2(T^*M)) = \cap_{k=1}^{\infty} C^k(\text{Sym}^2(T^*M))$ is a Fréchet space with respect to the family of C^k-norms.

Assume that M is compact. All Riemannian metrics are bounded and $\mathcal{M}$ is a subset of $C^\infty(\text{Sym}^2(T^*M))$. Equip $\mathcal{M}$ with the subspace topology. The space $\mathcal{M}$ is a convex cone in $C^\infty(\text{Sym}^2(T^*M))$ consisting of sections which are positive definite. Positive definiteness is an open condition and $\mathcal{M}$ is an open subset of $C^\infty(\text{Sym}^2(T^*M))$. Denote by $C^k\mathcal{M}$ the set of positive definite sections of $C^k(\text{Sym}^2(T^*V))$ and equip it with the subspace topology. Define

$$\mathcal{M}_k = L^2_k(\text{Sym}^2(T^*M)) \cap C^0\mathcal{M},$$

and equip it with the subspace topology of $L^2_k(\text{Sym}^2(T^*M))$.

Lemma 2.2.1. *Let (M^n, g) be a compact smooth manifold and let $k > n/2$ and $r \geq 0$. Then $\mathcal{M}_k$ is a Hilbert manifold, $C^r\mathcal{M}$ is a Banach manifold and $\mathcal{M}$ is a Fréchet manifold.*

Proof. For $k > n/2$ it follows from Theorem 2.1.5 that $L^2_k(\text{Sym}^2(T^*V))$ embeds continuously into $C^0(\text{Sym}^2(T^*V))$. the space $\mathcal{M}_k$ is the preimage of the open subset $C^0\mathcal{M}$ in $C^0(\text{Sym}^2(T^*V))$ under the continuous embedding, so the space $\mathcal{M}_k$ is an open subset of $L^2_k(\text{Sym}^2(T^*M))$. Hence it is a Hilbert manifold. The space $C^k\mathcal{M}$ is an open subset of the Banach space $C^k(\text{Sym}^2(T^*M))$, so it is a Banach manifold. The family of semi-norms on the space $C^\infty(\text{Sym}^2(T^*M)) = \cap_{k\geq 0} C^k(\text{Sym}^2(T^*M))$, which we get from the C^k-norms on $C^k(\text{Sym}^2(T^*M))$, turns it into a Fréchet space and as $\mathcal{M}$ is an open subset of it, $\mathcal{M}$ is a Fréchet manifold. □

2.3 Laplace operators

In this section we give a brief introduction to three Laplace operators. We also introduce some notation from differential geometry related to curvature and some relevant differential operators. The material is borrowed from [BES87, Chapter 1].

For a manifold (M, g) and a covariant derivative $\nabla : C^\infty(TM) \to C^\infty(TM \otimes T^*M)$.

The *Riemann curvature tensor* $R = R(\nabla) = F_\nabla \in \Omega^2_M(\text{End}(TM))$ is the $(3,1)$-tensor field $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. It can locally be written as $R = R_{ijk}{}^l\, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}$. The first two indices is the form part of R and the latter two the endomorphism part. For a diffeomorphism ϕ, R satisfies

$$R_{\phi^* g}(\phi^* X, \phi^* Y)\phi^* Z = \phi^*(R_g(X,Y)Z). \tag{2.1}$$

Locally ∇ can be written as $d_\nabla s = ds + As$, where A is the underlying connection of ∇ and ds is the exterior derivative applied componentwise to s. The curvature of ∇ can be written as $R(\nabla) = dA + A \wedge A$ and it measures the non-commutativity of ∇. A covariant derivative satisfying $R(\nabla) = 0$ is said to be *flat*. Metrics with a flat covariant derivative are locally isometric to Euclidean space. The *Ricci curvature tensor* is a symmetric $(2,0)$-tensor Ric $=$ Ric(g) defined as Ric$(X,Y) =$ tr$(V \mapsto R(X,V)Y)$. Locally it is given by Ric $= R_{ij} dx^i \otimes dx^j$ where $R_{ij} = R_{kij}{}^k$. The *scalar curvature* is the trace of the Ricci tensor. It is the function $s_g \in C^\infty(M)$ given by $s_g = \text{tr}_g(\text{Ric}) = \sum_{i,j=1}^n g^{ij} R_{ij}$. The *total scalar curvature* is $T_g = \int_M s_g \, dV_g$.

Assume that (M,g) is compact. The Levi-Civita Covariant derivative extends to a covariant derivative $\nabla : C^\infty(T^{(r,s)}M) \to C^\infty(T^{(r+1,s)}M)$. It admits a formal adjoint $\nabla^* : C^\infty(T^{(r+1,s)}M) \to C^\infty(T^{(r,s)}M)$ for the L^2-inner product. On $(k,0)$-tensors the operator ∇^* is locally given by

$$(\nabla^* \eta)(X_1, \ldots, X_k) = -\sum_{i=1}^n (\nabla_{e_i}\eta)(e_i, X_1, \ldots, X_k),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of TM and $X_1, \ldots, X_k$ are vector fields on M. The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ similarly admits a formal adjoint $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$. It is given by $d^*\omega = (-1)^{n(k+1)+1} *_g d *_g \omega$, where the *Hodge star operator* $*_g : \wedge^k(T_x^*M) \to \wedge^{n-k}(T_x^*M)$ satisfies $\alpha \wedge *\beta = (\alpha, \beta)_g \, dV_g$ for all $\alpha, \beta \in \wedge^k(T_x^*M)$. The operator d^* is the skew-symmetric part of ∇^*, i.e. $d^*\eta$ is the restriction of $\nabla^*\eta$ to $\wedge^{k+1}(M)$. Define the following operator on symmetric tensors:

$$\delta^* = \text{Sym} \circ \nabla|_{\text{Sym}^k(T^*M)} : C^\infty(\text{Sym}^k(T^*M)) \to C^\infty(\text{Sym}^{k+1}(T^*M)).$$

The formal adjoint of δ^* with respect to the L^2-inner product is

$$\delta = \nabla^*|_{\mathrm{Sym}^{k+1}(T^*M)} : C^\infty(\mathrm{Sym}^{k+1}(T^*M)) \to C^\infty(\mathrm{Sym}^k(T^*M)).$$

For $\alpha \in \Omega^1(M)$ the covariant derivative decomposes as $\nabla\alpha = \delta^*\alpha + \frac{1}{2}d\alpha$. The *Hessian* of a smooth function $f \in C^\infty(M)$ is the $(2,0)$-tensor field $\mathrm{Hess}(f) = \nabla(df)$ and it is often denoted $\nabla^2(f)$. The *divergence* of a vector field X is the smooth function $\mathrm{div}(X) = \delta(X^\flat) = -\sum_{i=1}^n g(\nabla_{e_i}X, e_i)$ where $\{e_i\}_{i=1}^n$ is an orthonormal basis.

Let $X \in C^\infty(TM)$ and let θ_t be the flow of X. The *Lie derivative* of a tensor field $A \in C^\infty(T^{(r,s)}M)$ in the direction of X is a map $\mathcal{L}_X : C^\infty(T^{(r,s)}M) \to C^\infty(T^{(r,s)}M)$ given by $(\mathcal{L}_X A)_p = \frac{d}{dt}|_{t=0}\, d(\theta_{-t})_{\theta_t(p)}(A_{\theta_t(p)})$. It satisfies $(\mathcal{L}_X A)_p = \frac{d}{dt}|_{t=0}((\theta_t)^*A)_p$, which means that Lie derivative of a tensor field A in the direction of X is the linearization of the pull-back action of the flow of X on A. We say that a vector field X is *Killing* if $\mathcal{L}_X g = 0$. This happens exactly when the local flow of X are all isometries.

We introduce three second order linear elliptic differential operators. The *Hodge Laplacian* $\Delta : \Omega^k(M) \to \Omega^k(M)$ is defined as

$$\Delta = dd^* + d^*d.$$

On functions, i.e. 0-forms, it takes the form $\Delta f = -\mathrm{tr}_g(Hess(f)) = -\mathrm{tr}_g(\nabla(df))$. The *rough Laplacian* is the operator

$$\nabla^*\nabla : C^\infty(T^{(r,s)}M) \to C^\infty(T^{(r,s)}M).$$

It satisfies $\nabla^*\nabla = -\mathrm{tr}(\nabla^2)$. The curvature tensor R acts on symmetric $(2,0)$-tensors as a linear map. We denote this action by $\overset{\circ}{R}$. It is defined as

$$(\overset{\circ}{R}\, h)(X,Y) = \sum_{i=1}^n h(R(X,e_i)Y, e_i), \tag{2.2}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of TM. The *Lichnerowicz Laplacian* is the differential operator $\Delta_L : C^\infty(T^{(k,0)}M) \to C^\infty(T^{(k,0)}M)$ given by $\Delta_L T = \nabla^*\nabla T + \Gamma T$, where ΓT is a 0'th order correction term. For the precise definition see [BES87, 1.143].

On Symmetric $(2,0)$-tensors the Lichnerowicz Laplacian is

$$\Delta_L h = \nabla^*\nabla h + 2\frac{s}{n}h - 2\overset{\circ}{R}h, \tag{2.3}$$

and for $\alpha \in \Omega^k(M)$, $\Delta_L\alpha = \Delta\alpha$. Also note that the Lichnerowicz Laplacian satisfies $\Delta_L(\operatorname{tr} h) = \operatorname{tr}(\Delta_L h)$. We shall see later that the Lichnerowicz Laplacian defines the first order deformations of Einstein metrics. For more about the Lichnerowicz Laplacian see [BER-EBI69]. The Hodge Laplacian and the rough Laplacian are related via the following *Weitzenböck formula*.

Lemma 2.3.1. *Let (M,g) be a compact connected Riemannian manifold. Then for any $\alpha \in \Omega^k(M)$,*

$$\Delta\alpha = \nabla^*\nabla\alpha + \tilde{R}\alpha$$

where $\tilde{R}\alpha$ is a smooth alternating $(k,0)$-tensor with coefficients

$$(\tilde{R}\alpha)_{r_1\ldots r_k} = \sum_{j=1}^{k} g^{ab}\mathit{Ric}_{r_j b}\alpha_{r_1\ldots a\ldots r_k} - 2\sum_{i<j} g^{bc}R^a_{r_i c r_j}\alpha_{r_1\ldots\ldots b\ldots r_k}.$$

On 1-forms the Weitzenböck formula takes the form

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha), \tag{2.4}$$

where $\mathrm{Ric}(\alpha)$ is the 1-form given by $\mathrm{Ric}(\alpha)(X) = \mathrm{Ric}(X,\alpha^\sharp)$.

2.4 Einstein metrics

In this section we give a brief introduction to Einstein metrics. For a comprehensive introduction to Einstein metrics see [BES87].

Let (M,g) be a connected Riemannian manifold. The metric g is *Einstein* if there exists some $\lambda \in C^\infty(M)$ for which

$$\mathrm{Ric}(g) = \lambda g \tag{2.5}$$

on all of M. An Einstein metric is ***Ricci-flat*** if $\lambda = 0$, i.e. $\text{Ric}(g) = 0$. Taking trace on both sides of (2.5) and remembering that $\text{tr}_g g = \text{tr}(g^{ij} g_{ij}) = n$, gives $s_g = \lambda n$. Equation (2.5) can therefore be rewritten as $\text{Ric}(g) = \frac{s_g}{n} g$. Applying $\text{tr} \circ \nabla$ on both sides of (2.5) and using the contracted Bianchi identity shows that the components of ∇s_g satisfy $\frac{1}{2}(\nabla s_g)^i = \frac{1}{n}(\nabla s_g)^i$. For $\dim(M) > 2$ Einstein metrics therefore have constant scalar curvature and constant Einstein coefficient λ.

Einstein metrics behave well under pull-backs by diffeomorphisms and rescaling by constants. From (2.1) it follows that

$$\phi^* \text{Ric}(g) = \text{Ric}(\phi^* g),$$

and from [BES87, Theorem 1.159] we know that

$$\text{Ric}(e^f g) = \text{Ric}(g) + (2-n)(\nabla df - df \otimes df) + (\Delta f - (n-2)||df||^2) g,$$

so rescaling g by a non-zero constant c satisfies $\text{Ric}(cg) = \text{Ric}(g)$.

Denote by $\mathcal{M}_1$ the subspace of all Riemannian metrics with $\text{vol}(g) = 1$. One reason why Einstein metrics could be considered 'distinguished' is [BES87, Theorem 4.21], which says that

Theorem 2.4.1. *Let M be a compact manifold. The critical points of the total scalar curvature functional*

$$T_g = \int_M s_g \, dV_g$$

restricted to $\mathcal{M}_1$ are exactly the Einstein metrics.

In physics, the total scalar curvature functional introduced in Theorem 2.4.1 is often named the Einstein-Hilbert action. It is usually denoted by S_g, but as we wish to keep that notation free for later use we have chosen the slightly unusual T_g for the total scalar curvature.

We make a few remarks about the existence of Einstein metrics on manifolds. It is convenient to consider 2, 3, 4 and ≥ 5 dimensional manifolds separately.

For 2-dimensional manifolds the Uniformization Theorem asserts that any connected oriented surface admits a complete metric with constant sectional curvature. On surfaces a metric is Einstein exactly if it has constant sectional curvature. So any connected and oriented 2-dimensional manifold therefore admits an Einstein metric.

For 3-dimensional manifolds a metric is Einstein exactly if it has constant sectional curvature. Contrary to the 2-dimensional case, not all 3-dimensional manifolds admit a constant sectional curvature metric. We give an example of one such manifold (from [BES87, 6.16]).

Example 2.4.2. *The manifold $S^1 \times S^2$ does not admit any Einstein metrics.*

We briefly sketch the argument for Example 2.4.2. Let (M, g) be a connected complete 3-manifold and denote by $(\widetilde{M}, \pi^* g)$ the universal cover of M. If (M, g) satisfies $\sec_g > 0$, then $(\widetilde{M}, \pi^* g)$ satisfies $\sec_{\pi^* g} > 0$. If $\widetilde{M}$ is compact, then $\widetilde{M} \simeq S^3$ by a theorem by Hamilton [HAM82]. If $\widetilde{M}$ is complete and non-compact, then $\widetilde{M} \simeq \mathbb{R}^3$ by the Cheeger-Gromoll-Meyer theorem. If (M, g) satisfies $\sec_g \leq 0$, then by the Cartan-Hadamard theorem the exponential map $\exp_g$ is a covering map and $\widetilde{M} \simeq \mathbb{R}^3$. The manifold $S^1 \times S^2$ has universal cover $\mathbb{R} \times S^2$, which is neither diffeomorphic to $\mathbb{R}^3$ or S^3 and it therefore does not admit a constant sectional curvature metric. In particular, it does not carry any Einstein metrics.

For 4-dimensional manifolds examples of manifolds that do not admit Einstein metrics exist. One such example is $S^1 \times S^3$, see [BES87, 6.32].

For n-dimensional manifolds with $n \geq 5$ no manifolds are known not to admit any Einstein metrics, [BES87, 0.21].

The *Einstein operator* is the map $E : \mathcal{M} \to C^\infty(\mathrm{Sym}^2(T^*M))$ given by

$$E(g) = \mathrm{Ric}(g) - \frac{s_g}{n} g. \tag{2.6}$$

A metric g is Einstein exactly if $E(g) = 0$. Einstein metrics on manifolds with $\dim(M) \geq 3$ have constant scalar curvature, so $T_g = s_g \int_M dV_g = s_g \,\mathrm{vol}(g)$. We can therefore express λ in terms of the total scalar curvature, $\lambda = \frac{s_g}{n} = \frac{T_g}{n \cdot \mathrm{vol}(g)}$. For compact M with $\dim(M) \geq 3$ we can equivalently express the Einstein operator as $E(g) = \mathrm{Ric}(g) - \frac{T_g}{n \cdot \mathrm{vol}(g)} g$. The Einstein operator is a non-linear second order differential operator on the space of metrics.

From [BES87, Theorem 1.174] we know that the linearization of $g \mapsto \text{Ric}(g)$ is

$$\begin{aligned}\text{Ric}'_g(h) &= \frac{1}{2}\Delta_L h - \delta_g^*\delta_g h - \frac{1}{2}\nabla_g d(\text{tr}_g h)\\ &= \nabla^*\nabla h - \delta_g^*\delta_g h - \frac{1}{2}\nabla_g d(\text{tr}_g h) - 2\,\mathring{R}\,h.\end{aligned}$$

Let g be an Einstein metric and let g_t be a smooth curve of metrics with $g_0 = g$. The linearized Einstein operator at g is

$$\begin{aligned}E'_g(h) &= \frac{d}{dt}E(g_t)|_{t=0} = \frac{d}{dt}(\text{Ric}(g_t) - \frac{S_{g_t}}{n}g_t)|_{t=0}\\ &= \text{Ric}'_g(h) - \left(\frac{(T_g)'}{n}g'_0 g + \frac{T_g}{n}g'_0\right) = \text{Ric}'_g(h) - \frac{T_g}{n}h\end{aligned}$$

where the last inequality follows because g is a critical point of the total scalar curvature functional. It leaves us with the following expression for the linearization of the Einstein operator,

$$E'_g(h) = \frac{1}{2}\Delta_L h - \delta_g^*(\delta_g)h - \frac{1}{2}\nabla_g d(\text{tr}_g h) - \frac{T_g}{n}h.$$

2.5 Kähler metrics

In this section we fix some notation from complex geometry. We introduce Kähler metrics and the first Chern class of a complex vector bundle and explain the relation between Ricci-forms and the first Chern class on Kähler manifolds. We also present Yau's solution to the Calabi conjecture. The material is borrowed from [JOY00] and [HUY05]. Another relevant reference is [YAU78].

A *complex manifold* of complex dimension m is a smooth manifold of real dimension $2m$ with a *holomorphic atlas*, i.e. an atlas such that each chart is homeomorphic to a polydisc in $\mathbb{C}^n$ and all transition functions are holomorphic. An *almost complex structure* J on a smooth manifold M is an endomorphism $J_p : T_pM \to T_pM$ satisfying $J_p^2 = -Id$ for each $p \in M$. An almost complex structure J is *integrable* if its Nijenhuis tensor N_J vanishes. An integrable almost complex structure is called a *complex structure*. A consequence of the Newlander-Nierenberg theorem is that we could equivalently have defined a complex manifold as a smooth manifold with an integrable almost complex structure. We write (M, J) for a complex manifold with a complex structure J. We will usually denote its complex dimension by m.

On a complex manifold (M, J) the complex linear extension of J to the complex tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ where $T^{1,0}M$ is the i'-th eingenspace of J and $T^{0,1}M$ is the $-i'$-eigenspace. The complex $(\wedge^k (T_{\mathbb{C}}M)^*, d_{\mathbb{C}})$ splits naturally as $\wedge^{p,q}M = \wedge^p (T^{1,0}M)^* \otimes_{\mathbb{C}} \wedge^q (T^{0,1}M)^*$ and we denote by $\mathcal{A}^{p,q}$ the sheaf of sections of $\wedge^{p,q}M$. The global sections $\mathcal{A}^{p,q}(M)$ are the forms of bidegree (p, q). Extending the exterior differential complex linearly to $TM \otimes \mathbb{C}$ provides a differential $d_{\mathbb{C}}$. The projections of $d_{\mathbb{C}}(\mathcal{A}^{p,q}(M))$ onto $\mathcal{A}^{p+1,0}(M)$ is denoted ∂ and the projection onto $\mathcal{A}^{0,q+1}(M)$ is $\bar{\partial}$. The holomorphic tangent bundle is $\mathcal{T}_M = T^{1,0}M$ and the dual bundle Ω_M. The sheaf of holomorphic k-forms is Ω^k_M for $0 \leq k \leq m$. The *canonical bundle* is the determinant bundle $K_M = \det(\Omega_M) = \Omega^m_M$. The (p, q)-Dolbeault cohomology is denoted $H^{p,q}(M) = H^{p,q}_{\bar{\partial}}(M)$. This construction extends to holomorphic vector bundles E and is denoted by $H^{p,q}(M, E)$.

A real $(1, 1)$-form β can be converted into a symmetric bilinear $(2, 0)$-form b via $b(X, Y) = \beta(JX, Y)$. We call β the *fundamental form* of b and we say that β is *positive*, *vanishing* or *negative* if the corresponding $(2, 0)$-form b is positive definite, vanishing or negative definite respectively. On a complex manifold (M, J) a Riemannian metric g is said to be *Hermitian* if it is compatible with J, i.e. $g(JX, JY) = g(X, Y)$.

Definition 2.5.1. *Let (M, J) be a complex manifold with a Hermitian metric g. We say that g is* Kähler *if its fundamental form ω is closed, i.e. $d\omega = 0$.*

Locally the Kähler condition translates to $\omega_{ij} = J^k_i g_{kj}$. We say that a complex manifold (X, J) is *Kählerienne*, if it admits Kähler metrics, even if we have not specified one. A Kähler form is non-degenerate, so ω^n is proportional to the volume form of g. The relation is $\omega^n = n!\, dV_g$.

Remark 2.5.2. *On a complex Hermitian manifold (M, J, g) with the Levi-Civita covariant derivative ∇ and where ω is the fundamental form of g, the Kähler condition can be expressed in various ways. We have*

$$g \text{ Kähler} \iff d\omega = 0 \iff \nabla\omega = 0 \iff \nabla J = 0.$$

The condition $\nabla J = 0$ can be rewritten as $\nabla_X JY = J\nabla_X Y$ for all $X, Y \in T_pM$ and all

$p \in M$.

Let (M, J) be a complex manifold and let $\pi : E \to M$ be a complex vector bundle with a covariant derivative ∇. Taking the trace on the endomorphism part of the Riemann curvature tensor produces a differential 2-form. It turns out that this form is closed, so it defines a class in cohomology. The cohomology class $[\frac{i}{2\pi}\text{tr}_{\mathbb{C}}(F_\nabla)] \in H^2(M, \mathbb{R})$ is therefore well-defined. For any other covariant derivative ∇' the forms $\text{tr}_{\mathbb{C}}(F_\nabla)$ and $\text{tr}_{\mathbb{C}}(F_{\nabla'})$ differ only by an exact form, so $[\frac{i}{2\pi}\text{tr}_{\mathbb{C}}(F_\nabla)]$ is independent of the choice of covariant derivative. This is the Chern-Weil construction of Chern classes.

Definition 2.5.3. *Let (M, J) be a complex manifold and let $\pi : E \to M$ be a complex vector bundle with a covariant derivative ∇. We define the* first Chern class *of ∇ on E to be*

$$c_1(E, \nabla) = \frac{i}{2\pi}[tr_{\mathbb{C}}(F_\nabla)] \in H^2(M, \mathbb{R}).$$

where the trace is taken in the endomorphism part of the curvature tensor.

Definition 2.5.4. *We define the first Chern class of a complex manifold (M, J) to be*

$$c_1(M) = c_1(\mathcal{T}_M) \in H^2(M, \mathbb{R})$$

where $\mathcal{T}_M$ is the holomorphic tangent bundle of M. The first Chern class depends on the choice of complex structure on X and we will therefore often denote it $c_1(J)$.

For any complex vector bundle we have $c_1(E) = c_1(\det(E))$. The canonical bundle K_M is the dual line bundle to the determinant bundle of $\mathcal{T}_M$, i.e. $K_M = -\det(\mathcal{T}_M)$. We can therefore express the first Chern class of M via the canonical line bundle as $c_1(M) = -c_1(K_M)$.

Similarly to the relation between the metric g and the Kähler form ω, where $\omega(u, v) = g(Ju, v)$, then the fundamental form of the Ricci curvature tensor $\text{Ric}(u, v)$ is the real $(1, 1)$-form $\rho = \rho(g) \in \mathcal{A}^{1,1}(M)$ defined by $\rho(u, v) = \text{Ric}(Ju, v)$, which is called the *Ricci-form* of g. Locally this relation is $\rho_{ij} = J_i^k r_{kj}$.

Proposition 2.5.5. *Let (M, J, g) be a Kähler manifold and let ρ be the Ricci-form of g. Then $\rho = i\, tr_{\mathbb{C}}(F_\nabla)$, where the trace is taken in the endomorphism part of the tensor.*

On a Kähler manifold we therefore have

$$c_1(J) = \frac{i}{2\pi}[\mathrm{tr}_{\mathbb{C}}(F_\nabla)] = \frac{1}{2\pi}[\rho] \in H^2(M, \mathbb{R}).$$

A Riemannian metric g on a complex manifold (M, J) is *Kähler-Einstein* if it is both Kähler and Einstein. For such metrics $\rho = \lambda\omega$, where ω and ρ are the Kähler form and Ricci form of g respectively and λ is the Einstein constant of g. On Kähler-Einstein manifolds we therefore have

$$c_1(J) = \frac{\lambda}{2\pi}[\omega] \in H^2(M, \mathbb{R}).$$

A cohomology class $\alpha \in H^2(M, \mathbb{R})$ has a *sign* if it contains a representative which has a sign, i.e. for which the associated symmetric $(2, 0)$-tensor has a sign. Cohomology classes may or may not have a sing. We write $\alpha < 0$, $\alpha = 0$ or $\alpha > 0$ for $\alpha \in H^2(M, \mathbb{R})$ in case α has a sign and the sign is negative, zero or positive respectively. We say that a complex vector bundle $\pi : E \to M$ has a sign if its first Chern class $c_1(E)$ has a sign, and we say that the sign of E is positive, zero or negative if $c_1(E)$ is positive, zero or negative respectively.

A metric g is Ricci-flat exactly when $\rho(g) = 0$, so if a manifold (M, J) admits a Ricci-flat metric which is Kähler with respect to J, then $c_1(J) = \frac{2\pi}{i}[\rho] = 0$. Ricci-flat Kähler metrics can therefore only exist on manifolds with vanishing first Chern class. If (M, J) is compact and g is Kähler-Einstein with Einstein constant λ and $c_1(J) = 0$, then this forces the Einstein constant of g to be zero as the volume of the metric is $0 \neq dV_g = \frac{\omega^n}{n!}$ so ω cannot be exact. Hence, $[\omega] \neq 0 \in H^2(M, \mathbb{R})$. But $0 = c_1(J) = \frac{\lambda}{2\pi}[\omega]$ which implies $\lambda = 0$.

The next celebrated result about the existence of Kähler metrics with prescribed Ricci form on compact Kähler manifolds was conjectured by Eugenio Calabi in the 1950's and proved by Shing-Tung Yau in the late 1970's ([YAU78]).

Theorem 2.5.6 (Calabi Conjecture)**.** *Let (M, J, g) be a compact complex Kähler manifold. Let ω be the Kähler form of g. For any $\rho' \in 2\pi c_1(J)$ there exists a Kähler metric g' with ρ' as its Ricci form and with Kähler form ω' satisfying $[\omega'] = [\omega]$.*

In the special case of vanishing first Chern class Theorem 2.5.6 can be used to construct Kähler-Einstein metrics with zero Einstein constant.

Corollary 2.5.7. *Let (M, J, g) be a compact complex Kähler manifold with $c_1(J) = 0$ and let ω be the Kähler form of g. Then there exists a Ricci-flat Kähler metric in the Kähler class of ω.*

2.6 Deformation theory of complex structures

In this section, we give an introduction to the deformation theory of complex structures on compact complex manifolds. Deformation theory of complex structures is often formulated using the language of complex spaces. For our purpose, however, a smooth version will suffice. The material is borrowed from [KOD86] and [HUY05]. Two other relevant sources are [KOD-NIR-SPE58] and [KOD-SPE60].

A map $\pi : X \to Y$ between topological spaces is *proper* if preimages of compact sets are compact. A smooth family of compact complex manifolds is defined as follows.

Definition 2.6.1. *Let $\mathcal{X}$ and S be complex manifolds and let $\pi : \mathcal{X} \to S$ be a proper holomorphic map. The fibres $\mathcal{X}_t = \pi^{-1}(t)$ are compact complex submanifolds of $\mathcal{X}$ and we say that $\pi : \mathcal{X} \to S$ is a* smooth family of complex manifolds *parametrized by S.*

Fixing $0 \in S$ and restricting $\pi : \mathcal{X} \to S$ to a germ around 0, the family $\pi : \mathcal{X} \to S$ from Definition 2.6.1 is a smooth family of deformations of the compact complex manifold $M = \mathcal{X}_0$. For such a family of deformations we can trivialize $\pi : \mathcal{X} \to S$ as a differentiable family. The fibres are all diffeomorphic and we can view $\mathcal{X} \simeq S \times M$ as a deformation of the complex structure on $M = \mathcal{X}_0$ parametrized by S. Elements of T_0S are called *infinitesimal complex deformations*.

Proposition 2.6.2. *Let M be a compact complex manifold and let S be the parameter space of a versal family of deformations of M. Then there is a natural bijection between the space of infinitesimal complex deformations T_0S and $H^1(M, \mathcal{T}_M)$.*

Next we define a particular class of compact complex manifolds which are well understood when it comes to complex deformations. This will be convenient when we start looking for applications of our results in Chapter 4.

Definition 2.6.3. *A* Calabi-Yau *manifold is a compact complex Kähler manifold with trivial canonical bundle.*

We say that a compact complex manifold M has *unobstructed deformations* if every infinitesimal deformation $v \in H^1(M, \mathcal{T}_M)$ integrates into a smooth curve of deformations. If $H^2(M, \mathcal{T}_M) = 0$, then M has unobstructed deformations. A special feature of a Calabi-Yau manifold M is that $H^2(M, \mathcal{T}_M) = 0$, and they therefore have unobstructed deformations.

In this thesis we will be studying properties of metrics and complex structures which are preserved under deformations. The next result by Kodaira will be important for our work.

Theorem 2.6.4 (Kodaira)**.** *Let M be a compact complex Kähler manifold. If $\pi : \mathcal{X} \to S$ is a smooth family of deformation of M, then $\mathcal{X}_t$ is Kähler for all $t \in S$.*

The next lemma is a stability result concerning the first Chern class $c_1(J)$.

Lemma 2.6.5. *Let (M, J) be a compact complex manifold and assume that $c_1(J) = 0$. Assume that M admits a smooth family of deformations $\pi : \mathcal{X} \to S$. Then for $t \in S$ small $c_1(J_t) = 0$.*

Lemma 2.6.5 is a consequence of the fact that the first Chern class takes values in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$, and it must therefore be preserved for small deformations. We say that a property which is preserved under small deformations is an *open condition*.

2.7 Koiso's deformation theory on compact manifolds

In this section, we give a review of Koiso's deformation theory of Einstein metrics over compact complex manifolds. Let (M, J, g) be a compact complex Kähler-Einstein manifold with negative or vanishing first Chern class and with unobstructed complex deformations. Koiso established in [KOI83] the remarkable fact that any Einstein deformation of g is Kähler, possibly with respect to a perturbed complex structure. He also showed that in a neighbourhood of a Kähler-Einstein structure the moduli space of Einstein structures is, up to an action of a finite group, a smooth manifold, and he found an expression for the dimension of it. We give an account of his results. For more details see [KOI83]. Alternatively see [BES87, Chapter 12] for an excellent review.

We focus on the case of vanishing first Chern class. Koiso's results also cover the case of negative first Chern class. The latter case is simpler than the former due to absence of any Hermitian infinitesimal Einstein deformations. Let (M, g) be a compact Ricci-flat manifold. We introduce an equivalence relation on the space of metrics $\mathcal{M}$. Identify two metrics in $\mathcal{M}$ if one is the pull-back of the other by a diffeomorphism or if it is a rescaling of the other by a positive constant. This gives an equivalence relation $g' \sim g$ if $g' = c\phi^* g$ for some $c > 0$ and some $\phi \in \mathcal{D}$, where $\mathcal{D} = \mathcal{D}(M)$ denote the group of diffeomorphisms from M to itself. The equivalence classes are called ***Riemannian structures*** and we denote the quotient space of Riemannian structures $\mathcal{M}/\sim$ by $\widetilde{\mathcal{M}}$. If we denote by $\mathcal{M}_1$ the space of metrics with $\text{vol}(g) = 1$, then we could equivalently describe the quotient space as $\mathcal{M}_1/\mathcal{D}$. Denote by $\text{Iso}(M, g)$ the group of isometries from (M, g) to itself. Ebin produced in [EBI70] a slice S_g for the action of $\mathcal{D}$ on $\mathcal{M}$ containing g. This construction produces a homeomorphism

$$U \subset \mathcal{M}_1/\mathcal{D} \to S_g/\text{Iso}(M, g), \tag{2.7}$$

where U is a neighbourhood of $[g]$. This serves as a chart for the quotient space. If g is Ricci-flat, then the structure $[g]$ is called a *Ricci-flat structure*. Denote the quotient space of Ricci-flat structures by $\widetilde{\mathcal{R}}$. . This is a subspace of $\widetilde{\mathcal{M}}$ and it is called the *moduli space* of Ricci-flat structures. Denote by P_g the subspace of Ricci-flat metrics in the slice S_g. It is called the *premoduli space* of Ricci-flat metrics.

In Section 2.3 we introduced the operator $\delta_g^* : \Omega^1(M) \to C^\infty(\text{Sym}^2(T^*M))$. By [BES87, Lemma 1.60] it satisfies $\delta_g^*\eta = \frac{1}{2}\mathcal{L}_{\eta^\#} g$. The L^2-formal adjoint of δ_g^* is $\delta_g : C^\infty(\text{Sym}^2(T^*M)) \to \Omega^1(M)$. The operator δ_g^* has finite dimensional kernel and closed image, so $C^\infty(\text{Sym}^2(T^*M))$ splits as a direct sum of the image of δ_g^* and the kernel of δ_g. The tangent space of $\mathcal{M}_1$ at g consist of symmetric 2-tensors h with $\int_M \text{tr}_g h \, d\text{vol}_g = 0$. Hence,

$$T_g\mathcal{M}_1 = \text{Im}(\delta_g^*) \oplus [\ker(\delta_g) \cap T_g\mathcal{M}_1] .$$

Let g_t be a smooth curve of Ricci-flat metrics with $g_0 = g$ and let $h = \frac{d}{dt} g(t)|_{t=0}$. The linearization of $\mathrm{Ric}(g_t)$ at g is

$$2\,\mathrm{Ric}'_g(h) = \nabla_g^*\nabla h - 2\delta^*\delta_g h - \nabla d(\mathrm{tr}_g h) - 2\overset{\circ}{R}\,h \tag{2.8}$$

where $\overset{\circ}{R}$ is the action of the Riemann curvature tensor on 2-tensors introduced in Section 2.3. Define by $\epsilon(g)$ the subspace of $C^\infty(\mathrm{Sym}^2(T^*M))$ satisfying the three equations

$$\mathrm{Ric}'_g(h) = 0 \qquad \delta_g h = 0 \qquad \int_M \mathrm{tr}_g(h)\,dV_g = 0. \tag{2.9}$$

Berger and Ebin simplified in [BER-EBI69] these equations to

$$(\nabla_g^*\nabla - 2\overset{\circ}{R})h = 0 \qquad \delta_g h = 0 \qquad \mathrm{tr}_g(h) = 0. \tag{2.10}$$

The first jet $h = \frac{d}{dt}g_t|_{t=0}$ of the family of Ricci-flat deformations g_t satisfies $h \in \epsilon(g)$, and elements of the space $\epsilon(g)$ is therefore called *infinitesimal Ricci-flat deformations*. An element $h \in \epsilon(g)$ is said to be *integrable* if there exists a smooth family of metrics through g with linearization h. The operator $\nabla^*\nabla - 2\overset{\circ}{R}$ is elliptic, so $\epsilon(g)$ is finite dimensional. Koiso showed that there exists a finite dimensional submanifold $Z \subset S_g$ with $T_g Z = \epsilon(g)$ and with P_g as a real analytic subset. All elements $h \in \epsilon(g)$ therefore integrate into a smooth curve of metrics in Z through g. Without further assumptions, then elements of $\epsilon(g)$ need not integrate into smooth curves of Ricci-flat metrics through g.

Assume that (M, J, g) is a compact complex Ricci-flat Kähler manifold. Symmetric 2-tensors $h \in C^\infty(\mathrm{Sym}^2(T^*M))$ split as $h = h_H + h_A$, where h_H is hermitian, i.e. $h_H(JX, JY) = h(X, Y)$, and h_A is skew-hermitian, i.e. $h_A(JX, JY) = -h_A(X, Y)$. From the tensor h we construct a new tensor $h \circ J(X, Y) = h(X, JY)$. Hermitian infinitesimal Ricci-flat deformations can be identified with real differential 2-forms of type $(1, 1)$ via the correspondence $h_H \mapsto h_H \circ J$. The equations (2.10) for $\epsilon(g)_H$ translate to

$$\begin{aligned} \mathrm{tr}_g\, h_H &= (h_H \circ J, \omega) \\ \delta_g h_H &= -\delta_g(h_H \circ J) \circ J \end{aligned}$$

and the usual real Laplacian Δ satisfies

$$\Delta(h_H \circ J) = (\nabla_g^* \nabla - 2 \overset{\circ}{R}) h_H \circ J.$$

Skew-hermitian infinitesimal Einstein deformations can be identified with infinitesimal complex deformations via $g \circ I = h_A \circ J$. They satisfy the following equations

$$\begin{aligned} \operatorname{tr}_g h_A &= 0 \\ \delta_g h_A &= -J \circ (\bar{\partial}^* I) \end{aligned}$$

and the complex Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ satisfies the Weitzenböck formula

$$g \circ (\Delta_{\bar{\partial}} I) = (\nabla^* \nabla - 2 \overset{\circ}{R}) h_A \circ J.$$

The operator $\nabla_g^* \nabla - 2 \overset{\circ}{R}$ preserves the Hermitian and skew-Hermitian type. This together with the above equations can be used to show that the Hermitian and skew-Hermitian splitting holds also for $\epsilon(g)$, i.e. for any $h \in \epsilon(g)$ both h_H and h_A belong to $\epsilon(g)$. Denote by $\epsilon(g)_H$ and $\epsilon(g)_A$ the vector subspaces of $\epsilon(g)$ of Hermitian and skew-Hermitian tensors respectively. If the first Chern class vanishes and g is a Ricci-flat Kähler metric, then the dimension of the space of infinitesimal Ricci-flat deformations is $\dim \epsilon(g) = \dim \epsilon(g)_H + \dim \epsilon(g)_A$ with $\dim \epsilon(g)_H = \dim H^{1,1}(M, \mathbb{R}) - 1$ and $\dim \epsilon(g)_A = 2 \dim_{\mathbb{C}} H^1(M, \mathcal{T}_M) - 2 \dim_{\mathbb{C}} H^{0,2}(M, \mathcal{T}_M)$. Koiso then showed that if all infinitesimal Ricci-flat deformations integrate into a smooth curves of Ricci-flat metrics through g, then the premoduli space P_g of Ricci-flat metrics in the slice S_g is a smooth manifold.

Assume that all infinitesimal complex deformations of (M, J) are integrable and denote by $\mathcal{J}$ the smooth parameter space of complex deformations of J. Denote by $\mathcal{V}$ the vector bundle over $\mathcal{J}$ for which $\mathcal{V}_t = \mathcal{H}^{1,1}_{J_t}(M, \mathbb{R})$ for each $t \in \mathcal{J}$. We need to show that all elements of $\epsilon(g)$ integrate into smooth curves of Ricci-flat Kähler metrics through g. Take $h \in \epsilon(g)$. To find a smooth curve of Ricci-flat Kähler metrics with first jet h we do the following. The tensor h can be related to an infinitesimal complex deformation $I \in H^1(M, \mathcal{T}_M)$. By assumption it integrates into a smooth curve of complex structures J_t. From Theorem 2.6.4 we know that for small t there exists a smooth curve of metrics g_t through g which are Kähler with respect to J_t. Let ω_t be the Kähler form $g \circ J_t$. It turns out that $t\kappa_t$ is an appropriate section of $\mathcal{V}$. It provides a smooth curve of Kähler forms $\omega_t + t\kappa_t$ with respect to J_t. Solutions to the complex Monge-Ampére equation,

$$(\omega_t + t\kappa_t + i\partial\bar{\partial}u)^m - Ae^f(\omega_t + t\kappa_t)^m = 0, \tag{2.11}$$

produces Ricci-flat Kähler metric $\tilde{g}_t$ of J_t in the class of $[\omega_t + t\kappa_t]$. Isolating f in (2.11) provides a map $F((J_t, \kappa_t), u) = f$. This map has a surjective differential so we can apply the implicit function theorem to obtain a smooth map ψ from a neighbourhood of $(0, 0)$ in $\mathcal{V}$ to the slice S_g whose image consists of Ricci-flat Kähler metrics. It turns out that the map ψ is surjective onto a neighbourhood of g in S_g, so Ricci-flat Kähler metrics span an entire neighbourhood of g in the slice S_g. The curve $\psi(J_t, \kappa_t)$ produces a smooth curve of Ricci-flat Kähler metrics in S_g through g with first jet h. The findings of Koiso can be summarized in the following two theorems.

Theorem 2.7.1. *Let (M, J) be a compact complex manifold with $c_1(J) = 0$. Let g be a Ricci-flat Käher metric on M and assume that all infinitesimal complex deformations are integrable. Then any Ricci-flat deformation of g is Kähler, possibly with respect to a perturbed complex structure.*

This can be used to get a local understanding of the moduli space or Ricci-flat structures $\widetilde{\mathcal{R}}$.

Theorem 2.7.2. *Let M be as in Theorem 2.7.1. Then in a neighbourhood of the Ricci-flat Kähler structure $[g]$ the moduli space of Ricci-flat structures is, up to an action of a finite group, a finite dimensional manifold of dimension*

$$\dim H^{1,1}_{\mathbb{R}}(M, J) - 1 + 2\dim_{\mathbb{C}} H^1(M, \mathcal{T}_M) - 2\dim_{\mathbb{C}} H^{0,2}(M, J).$$

Theorem 2.7.2 can be deduced from Theorem 2.7.1 via the following argument. The map from (2.7) produces a chart from a neighbourhood of $[g]$ in the moduli space of Ricci-flat structures $\widetilde{\mathcal{R}}$ onto $P_g/\mathrm{Iso}(M, g)$. Let $I(M, g)^0$ be the identity component of $\mathrm{Iso}(M, g)$. When the scalar curvature vanishes, then the identity component I_g^0 acts trivially on P_g. The isometry group $\mathrm{Iso}(M, g)$ is a compact Lie group and I_g^0 is a normal subgroup. The quotient $\mathrm{Iso}(M, g)/I_g^0$ is therefore a finite group. A model for the moduli space $\widetilde{\mathcal{R}}$ in a neighbourhood of a Kähler structure is therefore given by $P_g/(\mathrm{Iso}(M, g)/I_g^0)$. As P_g

spans an entire neighbourhood of g in the manifold Z, then $T_g P_g = \epsilon(g)$. The dimension of P_g is therefore $\dim \epsilon(g)_H + \dim \epsilon(g)_A$.

Chapter 3

Orbifold Ricci-flat deformations

3.1 Introduction and results

In this chapter, we generalize results by Koiso [KOI83] to orbifolds. In Section 2.7 we explained how Koiso in [KOI83] showed that if M is a compact complex manifold with vanishing first Chern class and with all infinitesimal complex deformations integrable, then Ricci-flat deformations of a Ricci-flat Kähler metrics on M are Kähler, possibly with respect to a perturbed complex structure. He also showed that the moduli space of Ricci-flat structures in a neighbourhood of a Ricci-flat Kähler structure is, up to the action of a finite group, a finite dimensional manifold. In this chapter we prove that a similar statement holds if you replace the compact manifold in Koiso's results with a compact orbifold. We also prove Ebin's slice theorem for compact orbifolds and the Calabi conjecture for compact Kähler orbifolds.

An orbifold (Definition 3.2.3) is a generalization of a manifold. The difference is that the local model on an orbifold is more complicated than on a manifold. Instead of being locally homeomorphic to an open subset of Euclidean space, orbifolds are locally homeomorphic to a quotient of an open subset of Euclidean space by an action of a finite group. This group may have non-trivial isotropy, which leads to the presence of singularities on an orbifold.

Orbifolds may be considered a natural class of objects as they appear in various construction on smooth manifolds. For instance, as we explained in Section 2.7, the moduli space of Einstein structures on a compact Kähler-Einstein manifold with non-positive first

Chern class and all infinitesimal complex deformations integrable also carries the structure of an orbifold. For other constructions involving orbifolds see for instance [BOY-GAL08].

For a compact complex Kähler-Einstein orbifold with vanishing first Chern class, we prove that integrability of all infinitesimal complex deformations implies that all infinitesimal Ricci-flat deformations integrate into smooth curves of Ricci-flat deformations. We use this to prove that a Ricci-flat deformation of a Ricci-flat Kähler metric is Kähler, possibly with respect to a perturbed complex structure. We also show that the moduli space of Ricci-flat structures is, up to an action of a finite group, a finite dimension manifold and we find an expression for the dimension of this moduli space in a neighbourhood of a Kähler structure.

To generalize results from manifolds to orbifolds there are in general two approaches. One is to use that the set of regular points V_{reg} in an orbifold V is a smooth manifold and that V_{reg} is a dense open subset. This way one can extend result from manifolds to orbifolds by continuity. The drawback of this approach is that the space V_{reg} is not compact. The other approach is to use that an orbifold locally is a quotient of a manifold by a finite group Γ. A problem on an orbifold can therefore locally be lifted to a Γ-invariant problem on a manifold. The drawback to the second method is that it is local in nature. A number of the tools we wish to make use of require compactness of the base space, so we will mostly employ the second method.

We now present the main results of this chapter. Proper definitions of the objects and operators involved will be given in the relevant sections. Let (V, J) be a complex orbifold and denote by $c_1(J)^{orb}$ the orbifold first Chern class in $H^2(V, \mathbb{R})$. If the orbifold has no singularities, i.e. is a manifold, then $c_1(J)^{orb}$ coincides with the usual first Chern class of the manifold and it takes values in the image of the integer cohomology in $H^2(V, \mathbb{R})$, however, in general $c_1(J)^{orb}$ takes values in the image of the cohomology with rational coefficients in $H^2(V, \mathbb{R})$. The Calabi conjecture is an important result about the existence of a Kähler metric with a prescribed Ricci-form on a compact Kähler manifold. In Section 3.5 we adapt the proof of the Calabi conjecture on compact Kähler manifolds from [JOY00, Chapter 5] to compact Kähler orbifolds.

Theorem 3.1.1. *Let (V, J, g) be a compact Kähler orbifold with Kähler form ω. Then for any real closed $(1,1)$-form $\rho' \in 2\pi c_1(J)^{orb}$ there exists a unique Kähler metric g' on (V, J) such that its Ricci-form is ρ' and its Kähler form ω' is cohomologous to ω.*

Let V be a compact orbifold and denote by $\mathcal{D}^{orb}$ the group of orbifold diffeomorphisms on V. For a Riemannian metric g on V denote by $\text{Iso}(V, g)^{orb}$ the subgroup of $\mathcal{D}^{orb}$ of isometries from (V, g) to itself. Denote by $\mathcal{M}^{orb}$ the space of orbifold Riemannian metrics on V. Similarly to the manifold case, we make use of the existence of a local slice in $\mathcal{M}$ for the action of $\mathcal{D}^{orb}$. This is an orbifold version of a similar result for compact manifolds proved by Ebin [EBI70]. Denote the action of $\eta \in \mathcal{D}^{orb}$ on $g \in \mathcal{M}^{orb}$ by $A(\eta, g) = \eta^* g$.

Theorem 3.1.2. *Let V be a compact orbifold. For each $g \in \mathcal{M}^{orb}$ there exists a submanifold $S = S_g \subseteq \mathcal{M}^{orb}$ containing g for which*

1. *If $\eta \in Iso(V, g)^{orb}$ then $A(\eta, S) = S$.*
2. *If $\eta \in \mathcal{D}^{orb}$ satisfies $A(\eta, S) \cap S \neq \emptyset$ then $\eta \in Iso(V, g)^{orb}$.*
3. *There exists a neighbourhood $U \subseteq \mathcal{D}^{orb}/Iso(V, g)^{orb}$ around the identity coset and a local section $\chi : U \to \mathcal{D}^{orb}$ such that*

$$F : U \times S \to \mathcal{M}^{orb} : (\eta, g) \mapsto A(\chi(\eta), g)$$

is a homeomorphism onto a neighbourhood of $g \in \mathcal{M}^{orb}$.

The following two theorems are generalizations of the results by Koiso from [KOI83] to compact orbifolds and are the main results of this chapter. The first result concerns the stability of the Kähler property for Ricci-flat deformations of a Ricci-flat Kähler metric. It is proved at the end of Section 3.7.

Theorem 3.1.3. *Let (V, J, g) be a connected compact complex orbifold with vanishing orbifold first Chern class and with all infinitesimal complex deformations integrable. Let g be a Ricci-flat Kähler metric on (V, J). Then small Ricci-flat deformations of g are Kähler, possibly with respect to a perturbed complex structure.*

Many results about compact manifolds have already been generalized to compact orbifolds, so Theorem 3.1.3 may come as less of a surprise to some. It has for instance previously been stated without proof by Boyer and Galicki in [BOY-GAL08, Theorem 5.5.5]. For a complex orbifold (V, J) denote by $\mathcal{T}_V$ the sheaf of holomorphic sections of the holomorphic tangent bundle. The second main result of this chapter is about the dimension of the moduli space of Ricci-flat structures in a neighbourhood of a Ricci-flat Kähler structure. It is proved in Section 3.8.

Theorem 3.1.4. *Assume the hypothesis of Theorem 3.1.3. In a neighbourhood of the Ricci-flat Kähler structure $[g]$ the moduli space of Ricci-flat structures is, up to an action of a finite group, a finite dimensional manifold of dimension*

$$\dim H^{1,1}_{\mathbb{R}}(V,J) - 1 + 2\dim_{\mathbb{C}} H^1(V,\mathcal{T}_V) - 2\dim_{\mathbb{C}} H^{0,2}(V,J).$$

The remaining part of Chapter 3 consists of seven sections. In Section 3.2, 3.3 and 3.4 we provide an introduction to orbifolds and various known results about them. In Section 3.5 we give a proof of the Calabi conjecture for compact Kähler orbifolds (Theorem 3.1.1). In Section 3.6 we prove Ebin's slice theorem for orbifolds (Theorem 3.1.2). In Section 3.7 and 3.8 we prove Theorem 3.1.3 and 3.1.4 respectively.

3.2 Orbifolds

Orbifolds were first introduced by Satake in [SAT56] and [SAT57], where he named them V-manifolds. They were later renamed orbifolds by Thurston [THU78]. An orbifold is a generalization of a manifold. Locally, it is modelled on a quotient of an open subset of Euclidean space by the action of a finite group. A good introduction to orbifolds can be found in [BOY-GAL08, Chapter 4]. In this section we present basic theory about orbifolds and orbifold bundles. The material is borrowed from the mentioned sources and [BAI56].

Definition 3.2.1. *Let V be a topological space and let $\bar{U} \subset V$ be an open subset. We say that a triple $\{U, \Gamma, \phi\}$ is a* local uniformizing system *or an (orbifold) chart for $\bar{U}$ if for some $n > 0$ it satisfies*

i) U is a connected neighbourhood of 0 in $\mathbb{R}^n$.

ii) $\Gamma \subset GL(n, \mathbb{R})$ is a finite subgroup, such that for all $\gamma \in \Gamma$, $\gamma(U) \subset U$ and the dimension of the set fixed by γ is less than or equal to $n-2$.

iii) the map $\phi : U \to \bar{U}$ is continuous and it satisfies $\phi(x) = \phi(\gamma.x)$ for all $\gamma \in \Gamma$ and induces a homeomorphism $\phi_\Gamma : U/\Gamma \to \bar{U}$.

Definition 3.2.2. *Let $\{U, \Gamma, \phi\}$ and $\{U', \Gamma', \phi'\}$ be local uniformizing systems. An* injection *is a smooth linear injective map $\lambda : U \to U'$ satisfying $\phi' \circ \lambda = \phi$.*

Definition 3.2.3. *An* orbifold atlas *on a Hausdorff and second countable topological space X is a family of orbifold charts $\{U_i, \Gamma_i, \phi_i\}$ with $X = \cup_i \phi_i(U_i)$ and such that for any $\{U_i, \Gamma_i, \phi_i\}$ and $\{U_j, \Gamma_j, \phi_j\}$ with some $x \in \phi_i(U_i) \cap \phi_j(U_j)$ there exist an orbifold chart $\{U_k, \Gamma_k, \phi_k\}$ with $x \in \phi_k(U_k)$ and injections $\lambda_{ki} : U_k \to U_i$ and $\lambda_{kj} : U_k \to U_j$. An atlas $\mathcal{F}$ is said to be a* refinement *of an orbifold atlas $\mathcal{G}$ if there exists an injection of every orbifold chart of $\mathcal{F}$ into some orbifold chart of $\mathcal{G}$. Two orbifold atlases are* equivalent *if they have a common refinement. An* orbifold *is a Hausdorff and second countable topological space equipped with an equivalence class of orbifold atlases.*

We will use the symbol V for an orbifold. We denote the collection of all local uniformizing systems on an orbifold V by $\mathcal{F}_V$ and the collection of all injections by $\mathcal{L}_V$. Observe that for a chart $\{U, \Gamma, \phi\}$ each $\gamma \in \Gamma$ is itself an injection. For a chart $\{U, \Gamma, \phi\}$ the *isotropy* subgroup of Γ at $x \in U$ is

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma.x = x\}.$$

For a chart $\{U, \Gamma, \phi\}$ we say that $x \in U$ is a *singular point* if Γ_x is non-trivial, i.e. $|\Gamma_x| > 1$. A *singularity* is a point $p \in \bar{U} \subset V$ for which $\phi(x) = p$ for a singular point $x \in U$. We denote the set of all singularities on V by V_{sing}. Points in $V \backslash V_{sing}$ are called *regular* and we denote them by V_{reg}. The space V_{reg} is an open dense subset of V.

For orbifolds V and W we define a *local orbifold map* to be a collection $\{f_U\}_{\{U,\Gamma,\phi\}\in\mathcal{F}_V}$ of maps such that for each $\{U, \Gamma, \phi\} \in \mathcal{F}_V$ there exists a $\{U^*, \Gamma^*, \phi^*\} \in \mathcal{F}_W$ with a map $f_U : U \to U^*$ satisfying that for each injection $\lambda \in \mathcal{L}_V$ with $\lambda : U \to U'$ there exists an injection $\lambda^* \in \mathcal{L}_W$ with $\lambda^* : U^* \to U^{*\prime}$ such that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{f_U} & U^* \\ \lambda \downarrow & & \downarrow \lambda^* \\ U' & \xrightarrow{f_{U'}} & U^{*\prime} \end{array}$$

The local orbifold map $\{f_U\}_{\mathcal{F}_V}$ gives rise to a map $f : V \to W$ satisfying $f \circ \phi = \phi^* \circ f_U$ for each chart $\{U, \Gamma, \phi\} \in \mathcal{F}_V$ (with the above notation). Such maps are called *orbifold maps* and we say they are of class C^k, L^p_k, C^∞ etc. if each local orbifold map is.

Definition 3.2.4. *Let V and E be orbifolds, let F be a smooth manifold and let $\pi : E \to V$ be a smooth map. Let G be a Lie group acting on F. We say (V, E, π, F, G), or simply*

$\pi : E \to V$, is an orbifold bundle if it satisfies the following conditions:

i) There is a one-to-one correspondence $\mathcal{F}_V \leftrightarrow \mathcal{F}_E$ such that if $\{U, \Gamma, \phi\}$ corresponds to $\{U^*, \Gamma^*, \phi^*\}$ then $U^* = U \times F$ and if we denote the projection $U^* \to U$ by π_{U^*}, then $\pi \circ \phi^* = \phi \circ \pi_{U^*}$.

ii) Let $\{U, \Gamma, \phi\} \leftrightarrow \{U^*, \Gamma^*, \phi^*\}$ and $\{U', \Gamma', \phi'\} \leftrightarrow \{U^{*'}, \Gamma^{*'}, \phi^{*'}\}$ with $\phi(U) \subseteq \phi(U')$. Then $\phi^*(U^*) \subseteq \phi^{*'}(U^{*'})$ and there exists a one-to-one correspondence between injections $\lambda : U \to U'$ and $\lambda^* : U^* \to U^{*'}$ such that for $(p, q) \in U^* = U \times F$ we have $\lambda^*(p, q) = (\lambda(p), \sigma_\lambda(p)q)$ where $\sigma_\lambda : U \to G$ is a smooth map satisfying $\sigma_{\mu\lambda}(p) = \sigma_\mu(\lambda(p))\sigma_\lambda(p)$ for injections $\lambda : U \to U'$ and $\mu : U' \to U''$.

We call the collection (V, E, π, F, G) for an orbifold bundle.

A *section* of an orbifold bundle (V, E, π, F, G) is an orbifold map $\{s_U\}$ from V to E such that each $s_U : U \to U \times F$ is a section in the usual sense and such that for each injection $\lambda \in \mathcal{L}_V$ with $\lambda : U \to U'$ we have, using the notation from Definition 3.2.4, that $s_{U'} \circ \lambda = \lambda^* \circ s_U$. We denote the space of for example smooth sections of an orbifold bundle E by $C^\infty(E)^{orb}$.

For an orbifold V we fix coordinates $\{u^1, \ldots, u^n\}$ for each chart $\{U, \Gamma, \phi\}$. Let $F = \mathbb{R}^n$ and let $G = \mathrm{GL}(n, \mathbb{R})$. For each injection $\lambda : U \to U'$ let σ_λ be the Jacobian matrix of λ at p, i.e.

$$\sigma_\lambda(p) = \left(\frac{\partial\, u'^i \circ \lambda}{\partial u^j} \right).$$

[SAT57, Theorem 1] then tells us that $(V, E, \pi, \mathbb{R}^n, \mathrm{GL}(n, \mathbb{R}))$ is an orbifold bundle. We call this bundle the *tangent bundle of V*.

More generally, we can construct an (r, s)*-orbifold tensor bundle* by setting

$$\sigma_\lambda(p) = \underbrace{\left(\frac{\partial u'^i}{\partial u^j} \right) \times \ldots \times}_{r} \underbrace{\left(\frac{\partial u^j}{\partial u'^i} \right) \times \ldots}_{s}$$

where $\times$ denotes the Kronecker product of matrices and where we set $F = \mathbb{R}^{n(r+s)}$ and let $G = \mathrm{GL}(n, \mathbb{R})$ operate on F as an (r, s)-tensor representation. We denote by $T^{(r,s)}V$ the (r, s)-tensor bundle on V.

For a chart $\{U, \Gamma, \phi\}$ we denote by $\Omega^k(U)$ the usual differential k-forms and by $\Omega^k(U)_\Gamma$ the subset of differential k-forms ω_U on U satisfying

$$(\omega_U)_p((p, X_1), \ldots, (p, X_k)) = (\omega_U)_{\gamma.p}((\gamma.p, \sigma_\gamma(p)X_1), \ldots, (\gamma.p, \sigma_\gamma(p)X_k)),$$

i.e. the differential k-forms ω_U is invariant under the action of Γ. We say that $\omega = \{\omega_U\}_{\mathcal{F}_V}$ is an orbifold differential k-form on V if for each chart $\{U, \Gamma, \phi\}$ the form ω_U is in $\Omega^k(U)_\Gamma$ and for each injection $\lambda : U \to U'$ the form ω_U satisfies

$$(\omega_U)_p(X_1, \ldots, X_k) = (\omega_{U'})_{\lambda(p)}(\sigma_\lambda(p)X_1, \ldots, \sigma_\lambda(p)X_k).$$

We denote the space of *orbifold differential k-forms* on V by $\Omega^k(V)^{orb}$.

For a $(k, 0)$-tensor $\omega = \{\omega_U\}$ the tensors $\mathrm{Sym}(\omega_U)$ and $\mathrm{Alt}(\omega_U)$ are Γ-invariant and compatible with injections, so they are well-defined operations on orbifolds. Similarly, the tensor product, wedge product and symmetric product are well-defined on orbifolds. The usual exterior differential operator d preserves Γ-invariance and is compatible with injections, so $d : \Omega^p(U)_\Gamma \to \Omega^{p+1}(U)_\Gamma$ lifts to define an exterior differential on $\Omega^*(V)^{orb}$. Orbifolds admit Riemannian metrics g and volume forms. A *Riemannian metric* on V is an element $g = \{g_U\}_{\mathcal{F}(V)} \in C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$ such that for each orbifold chart $\{U, \Gamma, \phi\}$ the g_U is a Γ-invariant Riemannian metric and such that each injection $\lambda : \{U, \Gamma, \phi\} \to \{U', \Gamma', \phi'\}$ is an isometry, i.e. $\lambda^* g_{U'} = g_U$. The last statement is equivalent to a reduction of the structure group of V from $GL(n, \mathbb{R})$ to $O(n)$. We denote the volume form of g on V by dV_g^{orb}.

An orbifold V is *orientable* if the coordinate systems of its charts can be chosen consistently such that $\det\left(\frac{\partial u'^i \circ \lambda}{\partial u^j}\right) > 0$ for each injection $\lambda \in \mathcal{F}_V$. This is equivalent to a reduction of the structure group to $SO(n)$.

Many differential geometric constructions are naturally compatible with the action of the isotropy groups, and the definition of orbifolds in general, for the following simple reason: For a chart $\{U, \Gamma, \phi\}$ let f be a smooth function on U. The tangent space $(T_pU)_\Gamma$ at p consist of those $X \in T_pU$ which are invariant under the action of the Jacobian $D\gamma_p$ for all $\gamma \in \Gamma$. If f satisfies $f(\gamma.p) = f(p)$ then $D(f \circ \gamma)_p(X) = Df_{\gamma.p} \circ D\gamma_p(X) = Df_p(X)$.

For an orientable orbifold V define the integral $\int_V \eta$ of an n-form η in the following way. If the closure of the set $\{p \in V \mid \eta_p \neq 0\}$ is contained in $\phi(U)$ for a chart $\{U, \Gamma, \phi\}$, set $\int_V \eta := \frac{1}{N_\Gamma} \int_V \eta_U$, where N_Γ is the order of the isotropy group in Γ. In general, define the *integral* of an n-form η to be $\int_V \eta = \sum_i \int_V f_i \eta$, where $\{f_i\}$ is a partition of unity satisfying that each f_i has support inside $\phi(U)$ for some chart $\{U, \Gamma, \phi\}$. Similarly, for a Riemannian orbifold (V, g) we define the integral of a smooth function $f : V \to \mathbb{R}$ by $\int_V f \, dV_g^{orb}$. We denote the volume of a Riemannian orbifold (V, g) by $\mathrm{vol}(g)^{orb}$. As remarked in [BOY-GAL08, p. 113], Stokes theorem holds on orbifolds.

We denote by $H^*_{dR}(V)$ the de Rham cohomology of $\Omega^*(V)^{orb}$. Satake proved in [SAT56, Theorem 1] that the de Rham cohomology on an orbifold V computes the usual cohomology with real coefficients, i.e. for all $0 \leq k \leq \dim(V)$ we have $H^k_{dR}(V) \simeq H^k(V, \mathbb{R})$. In some way this shows that $H^*_{dR}(V)$ is not a suitable cohomology theory for the study of orbifolds, as it does not 'see' the isotropy. For the purpose of this project it will suffice though. For other approaches see [BOY-GAL08, p. 117-118]

We will end our introduction to the basic concepts here. But many more differential geometric constructions generalize to orbifolds. For instance the construction of connections, covariant derivatives and curvature generalizes to orbifolds. So does the existence of Levi-Civita covariant derivatives, the Bianchi identity and Hodge decomposition. See [BAI56] for details.

3.3 Tools from analysis

In this section we introduce basic analysis on orbifolds.

Let (V, g) be a compact oriented Riemannian orbifold. Denote by $L^p(V)^{orb}$ the space of integrable orbifold maps $f : V \to \mathbb{R}$ with finite L^p-norm, $||f||_{L^p} = \left(\int_V |f|^p dV_g^{orb}\right)^{1/p}$. Define the *Sobolev space* $L^p_k(V)^{orb}$ to be the set of those L^p-orbifold functions which are k-times weakly differentiable such that $|\nabla^r f| \in L^p(V)^{orb}$ for $0 \leq r \leq k$. We equip $L^p_k(V)^{orb}$ with the Sobolev norm $||f||_{L^p_k} = \left(\sum_{i=0}^k \int_V |\nabla^i f|^p dV_g^{orb}\right)^{1/p}$. For $k \geq 0$, denote by $C^k(V)^{orb}$ the space of bounded and continuous functions from V to $\mathbb{R}$ which have k-times continuous bounded derivatives. We equip it with the norm $||f||_{C^k} = \sum_{i=0}^k \sup_V |\nabla^i f|$. Let $\alpha \in (0, 1)$. We say that a function $V \to \mathbb{R}$ is Hölder continuous with exponent α if $[f]_\alpha = \sup_{x \neq y \in V} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty$. We define the *Hölder space* $C^{k,\alpha}(V)^{orb}$ as the space of those C^k-functions for which $[\nabla^i f]_\alpha$ exists and is finite for each $i \leq k$. The four constructions $L^p(V)^{orb}$, $L^p_k(V)^{orb}$, $C^k(V)^{orb}$, and $C^{k,\alpha}(V)^{orb}$ generalize to the space of sections of an orbifold vector bundle (V, E, π, F, G) in the same way as on manifolds. We denote the corresponding spaces of sections by $L^p(E)^{orb}$, $L^p_k(E)^{orb}$, $C^k(E)^{orb}$, and $C^{k,\alpha}(E)^{orb}$.

Theorem 3.3.1 (Sobolev Embedding Theorem)**.** *Let (V, E, π, F, G) be an orbifold vector bundle over a compact orbifold V with $\dim(V) = n$. For all $l > \frac{n}{2} + s$ there exists a continuous linear embedding $L^2_l(E)^{orb} \to C^s(E)^{orb}$.*

Theorem 3.3.2 (Kondrashov's Theorem)**.** *Let (V, g) be a compact n-dimensional orbifold. Let k, l be integers with $k \geq l \geq 0$. Let $q, r \geq 1$ be real numbers and let $\alpha \in (0, 1)$. If*

$$\frac{1}{q} < \frac{1}{r} + \frac{k-l}{n}$$

then the embedding $L^q_k(V)^{orb} \to L^k_l(V)^{orb}$ is compact. If

$$\frac{1}{q} < \frac{k-l-\alpha}{n}$$

then $L^q_k(V)^{orb} \to C^{l,\alpha}(V)^{orb}$ is compact. Also $C^{k,\alpha}(V)^{orb} \to C^k(V)^{orb}$ is compact.

We remark that while orbifolds admit singularities, then the spaces of sections of orbifold vector bundles do not. They are vector spaces just as on manifolds. Though not a very profound observation, then it is nevertheless central. It ensures that the analysis on the spaces of sections of orbifold bundles is very similar to that of manifolds. We see an immediate consequence of this in the next proposition.

Proposition 3.3.3. *Let (V, g) be a compact orbifold and let $k \geq 0$, $p \geq 1$ and $\alpha \in (0, 1)$, then $L^2_k(E)^{orb}$ is a Hilbert space. $L^p_k(E)^{orb}$, $C^k(E)^{orb}$ and $C^{k,\alpha}(E)^{orb}$ are Banach spaces and $C^\infty(E)^{orb}$ is a Fréchet space.*

Proof. The spaces in question are vector spaces under the usual addition $f + g = \{f_U + g_U\}$, where $\{f_U\}$ and $\{g_U\}$ are the local orbifold maps of f and g respectively. First consider $C^k(E)^{orb}$. Let $f_i = \{(f_i)_U\}$ be a Cauchy sequence in $C^k(E)^{orb}$. For each l.u.s. $\{U, \Gamma, \phi\}$ the space $C^k(U, E)$ is a Banach space so the sequence $\{(f_i)_U\}$ converges to a section $f_U \in C^k(U, E)$. For $\gamma \in \Gamma$, the function $||(f_i)_U(x) - (f_i)_U(\gamma.x)||_{C^k}$ is continuous in x, so $||f_U(x) - f_U(\gamma.x)||_{C^k} = \lim_{i\to\infty} ||(f_i)_U(x) - (f_i)_U(\gamma.x)||_{C^k} = 0$. Hence $f_U \in C^k(U, E)_\Gamma$ for each l.u.s. and each $\gamma \in \Gamma$, so $f = \{f_U\} \in C^k(E)^{orb}$.

The same argument shows that $C^{k,\alpha}(E)^{orb}$ and $L^2_k(E)^{orb}$ are Banach spaces with their respective norms. Furthermore the norm on $L^2_k(E)^{orb}$ is generated by the inner product $(f,g) = \sum_{|\alpha|\leq k}(\nabla^\alpha f, \nabla^\alpha g)_{L^2(E)^{orb}}$. Equipping $C^\infty(E)^{orb}$ with the family of semi-norms from each $C^k(E)^{orb}$ turns it into a Fréchet space. □

We define the *linearization* of a differential operator P on orbifolds in the same way as on manifolds.

$$L_u P(v) = \lim_{t\to 0} \frac{P(u+tv) - P(u)}{t}.$$

The same argument as in Proposition 3.3.3 shows that $L_u P$ is well defined on orbifolds. The next three results are central for the study of elliptic operators. The first one is elliptic regularity and the third one is the Fredholm alternative.

Theorem 3.3.4. *Let (V,g) be a compact orbifold and let E_1, E_2 be orbifold vector bundles over V of the same dimension. Let P be a smooth linear elliptic differential operator of order k from E_1 to E_2. Let $\alpha \in (0,1)$ and let $l \geq 0$. Assume that we have $u \in L^1(E_1)^{orb}$ and $v \in L^1(E_2)^{orb}$ such that $Pu = v$. If $v \in C^\infty(E_2)^{orb}$ then $u \in C^\infty(E_1)^{orb}$. If $v \in C^{l,\alpha}(E_2)^{orb}$ then $u \in C^{k+l,\alpha}(E_1)^{orb}$ and*

$$||u||_{C^{k+l,\alpha}} \leq C(||v||_{C^{l,\alpha}} + ||u||_{C^0}).$$

for some $C > 0$ independent of v and w.

Proof. Direct adaptation of the proof of [JOY00, Theorem 1.4.1]. □

Theorem 3.3.5. *Let E_1, E_2 be orbifold vector bundles over a compact orbifold V, and let P be a smooth linear elliptic operator of order k from E_1 to E_2. Then P acts by $P : C^\infty(E_1)^{orb} \to C^\infty(E_2)^{orb}$, $P : C^{k+l,\alpha}(E_1)^{orb} \to C^{l,\alpha}(E_2)^{orb}$ and $P : L^p_{k+l}(E_1)^{orb} \to L^p_l(E_2)^{orb}$. Then kernel $\ker(P)$ of P is the same for all of these actions, and it is a finite-dimensional vector subspace of $C^\infty(E_1)^{orb}$.*

Proof. Direct adaptation of the proof of [JOY00, Theorem 1.5.1]. □

Theorem 3.3.6. *Let (V,g) be a compact orbifold and let E_1, E_2 be orbifold vector bundles over V, equipped with metrics in the fibres, and P is a smooth linear elliptic operator of order k from E_1 to E_2. Let $l \geq 0$ be an integer and let $p > 1$, and let $\alpha \in (0,1)$. Then the* *images of the maps* $P : C^{k+l,\alpha}(E_1)^{orb} \to C^{l,\alpha}(E_2)^{orb}$ *and* $P : L^p_{k+l}(E_1)^{orb} \to L^p_l(E_2)^{orb}$ *are closed linear subspaces of* $C^{l,\alpha}(E_2)^{orb}$ *and* $L^p_l(E_2)^{orb}$ *respectively. If* $v \in C^{l,\alpha}(E_2)^{orb}$ *then there exists* $u \in C^{k+l,\alpha}(E_1)^{orb}$ *with* $Pu = v$ *if and only if* $v \perp \ker(P^*)$*, and if one requires that* $u \perp \ker(P)$ *then* u *is unique. Similarly, if* $v \in L^p_l(E_2)^{orb}$ *then there exists* $u \in L^p_{l+k}(E_1)^{orb}$ *with* $Pu = v$ *if and only if* $v \perp \ker(P^*)$*, and if* $u \perp \ker(P)$ *then* u *is unique.*

Proof. Direct adaptation of the proof of [JOY00, Theorem 1.5.3]. □

3.4 Complex orbifolds

In this section, we introduce concepts related to complex orbifolds. The material is either borrowed from [BAI56], [BOY-GAL08, Chapter 4] or adapted to orbifolds from [HUY05, Chapter 6].

We define a *complex orbifold* (V, J) to be an even dimensional real orbifold of real dimension $2m$ with local charts $\{U, \Gamma, \phi\}$ where $U \subset \mathbb{C}^m$ and Γ is a finite subgroup of $GL(m, \mathbb{C})$ such that J_U is a complex structure on U compatible with Γ and $J = \{J_U\}_{\mathcal{F}(V)}$ is compatible with injections. The construction of Dolbeault cohomology generalizes to orbifolds. For a l.u.s. $\{U, \Gamma, \phi\}$ denote by $\mathcal{A}^*_{J_U}(U)_\Gamma$ the Γ-invariant complex differential forms on U. Denote by $\mathcal{A}^*_J(V)^{orb}$ the complex differential forms on (V, J). The (p, q)-Dolbeault cohomology group is denoted by $H^{p,q}(V)$. A *Kähler* metric g on (V, J) is an orbifold Riemannian metric for which the fundamental form $\omega(X, Y) = g(JX, Y)$ is d-closed. The presence of a Kähler metric on (V, J) is equivalent to a reduction of the structure group to $U(m)$. Baily proved in [BAI56] the Hodge decomposition theorem for orbifolds. From this it follows that the $\partial\bar{\partial}$*-Lemma* extends to orbifolds.

Lemma 3.4.1 (Global $\partial\bar{\partial}$-Lemma)**.** *Let* (V, J, g) *be a compact Kähler orbifold and let* η *be a smooth exact real* $(1, 1)$*-form on* V*. Then there exists an* $f \in C^\infty(V)^{orb}$ *such that* $\eta = i\partial\bar{\partial} f$*.*

We can give the orbifold V the structure of a ringed space $(V, \mathcal{O}_V)$ over $\mathbb{C}$ in the following way. For $x \in V$ take a chart $\{U, \Gamma, \phi\}$ for which $x \in \phi(U)$. The stalk $\mathcal{O}_x$ is isomorphic to the local ring of germs of Γ-invariant holomorphic functions on U. Let $\mathcal{O}_V$ be the structure sheaf with these stalks. Following [MOE-PRO97, Section 2] we define an *orbifold sheaf*, or *orbisheaf*, to be a sheaf $\mathcal{F}$ on the orbifold V satisfying i) for each chart $\{U, \Gamma, \phi\}$, $\mathcal{F}_U$ is a sheaf on U and ii) for each injection $\lambda : \{U, \Gamma, \phi\} \to \{U', \Gamma', \phi'\}$ there exists an isomorphism of sheaves $\mathcal{F}_\lambda : \mathcal{F}_U \to \lambda^* \mathcal{F}_{U'}$. The construction of sheaf cohomology for orbisheaves goes through on orbifolds and for an orbisheaf $\mathcal{F}$ we denote by $H^*(V, \mathcal{F})$ the orbifold sheaf cohomology of it. Denote by $\mathcal{T}_V$ the orbisheaf of sections of the holomorphic tangent bundle $T^{1,0}V$. We define the *structure orbisheaf* $\mathcal{O}_V^{orb}$ of the orbifold V to be the orbisheaf defined by the structure sheaf $\mathcal{O}_U$ for each chart $\{U, \Gamma, \phi\}$. By [BOY-GAL08, Proposition 4.2.18] isomorphism classes of orbifold vector bundles and locally free orbisheaves are in a one-to-one correspondence. The *canonical orbisheaf* K_V^{orb} is defined as the orbisheaf $\det(\Omega_V) = \Omega_V^m$. The Dolbeault cohomology $H^{p,q}(V)$ computes the sheaf cohomology of the sheaf Ω_V^p, i.e. $H^{p,q}(V) \simeq H^q(V, \Omega_V^p)$.

On a complex manifold M the first Chern class of a complex vector bundle $\pi : E \to M$ is a class $[\frac{i}{2\pi}\mathrm{tr}_{\mathbb{C}}(F_\nabla)] \in H^2(M, \mathbb{R})$ for a covariant derivative ∇ on E. This class is independent of the choice of covariant derivative and is an invariant of the complex structure on E. The first Chern class of M is the first Chern class of the holomorphic tangent bundle $T^{1,0}M$. The class $c_1(M, J)$ is in the image of integer cohomology. This Chern-Weil construction of Chern classes goes through on orbifolds just as on manifolds, and we have

Definition 3.4.2. *Let (V, J) be a complex orbifold and $\pi : E \to V$ a complex vector bundle. Let ∇ be a covariant derivative on E. Define the* orbifold first Chern class *of ∇ on E to be*

$$c_1(E, \nabla)^{orb} = [\frac{i}{2\pi} tr_{\mathbb{C}}(F_\nabla)] \in H^2_{dR}(V) \simeq H^2(V, \mathbb{R}).$$

Define the orbifold first Chern class *of (V, J) to be $c_1(T^{1,0}V)^{orb}$. It only depends on the complex structure J, so we denote it by $c_1(J)^{orb}$.*

In [BOY-GAL08, Section 4.4] they explain the interplay between the orbifold first Chern class $c_1(J)^{orb}$ and the singularities on a complex orbifold V. In this thesis we will only use $c_1(V)^{orb}$ constructed via Chern-Weil theory so it is not essential for our work, but as many differential geometric constructions extend naturally to orbifolds, we thought it would be appropriate to at least briefly mention the relation outlined in [BOY-GAL08, Section 4.4]. It is an example of a generalization to orbifolds which requires some more care.

A *Baily divisor* is a collection of divisors $\{D_U\}_{\mathcal{F}(U)}$ such that for each chart $\{U, \Gamma, \phi\}$, D_U is a Cartier divisor on U satisfying $i)$ if for each $x \in V$ and each $\gamma \in \Gamma$, $f \in \mathcal{D}_{\gamma x}$ then $f \circ \gamma \in \mathcal{D}_x$ and $ii)$ if $\lambda : U \to U'$ is an injection and $f \in \mathcal{D}'_{\lambda x}$ then $f \circ \lambda \in \mathcal{D}_x$. Here $\mathcal{D}$ is the divisor sheaf of the Baily divisor and $\mathcal{D}_x$ is the stalk of $\mathcal{D}$ at x. A *branch divisor* is a Weil divisor on V with coefficients in $\mathbb{Q}$ of the form $\sum_\iota \left(1 - \frac{1}{m_\iota}\right) D_\iota$ where the sum is taken over all Weil divisors D_ι in V_{sing} and $m_\alpha = gcd\{|\Gamma_x|\}_{x \in D_\alpha}$ is the *ramification index* of D_α. By [BOY-GAL08, Proposition 4.4.13] all branch divisors lift to a Baily divisor via $\phi^* D_\alpha$ for each chart $\{U, \Gamma, \phi\}$. Define a *canonical divisor* D_V to be any divisor on V such that the line bundle of $D_V \cap V_{reg}$ is the canonical bundle $K_{V_{reg}}$. If $V_{sing} = \emptyset$ the line bundle L_{D_V} of D_V is the canonical bundle K_V. If $V_{sing} \neq \emptyset$ then K_V is in general not defined. By [BOY-GAL08, Proposition 4.4.15] the canonical divisor D_V^{orb} is related to the divisor D_V via $D_V^{orb} = \phi^* D_V + \sum_\alpha \left(1 + \frac{1}{m_\alpha}\right) \phi^* D_\alpha$, on each chart $\{U, \Gamma, \phi\}$. The first Chern class of a Baily divisor is defined as the first Chern class of the corresponding complex line bundle. The first Chern class $c_1(J)^{orb}$ is not in the image of the inclusion $H^*(V, \mathbb{Z}) \to H^*(V, \mathbb{R})$, but instead $c_1^{orb}(J)$ is in the image of $H^*(V, \mathbb{Q}) \to H^*(V, \mathbb{R})$ ([BOY-GAL08, p. 120-121]) and by [BOY-GAL08, (4.4.2)] the orbifold first Chern class is related to the first Chern class $c_1(L_{D_\mathbf{V}})$ via

$$c_1(V)^{orb} = c_1(L_{D_V}) - \sum_\alpha \left(1 - \frac{1}{m_\alpha}\right) c_1(L_{D_\alpha}). \tag{3.1}$$

It follows from (3.1) that if the orbifold V does not admit any branch divisors in its singular locus, then $c_1(V)^{orb}$ coincide with $c_1(L_{D_V})$ and if $V_{sing} = \emptyset$ then $c_1(L_{D_V})$ recovers the usual first Chern class of the smooth manifold V.

On a complex orbifold (V, J, g) the *fundamental form* of a J-invariant symmetric bilinear tensor h is the real $(1,1)$-form $\psi(X, Y) = h(JX, Y)$. The fundamental form of the Ricci-tensor $\mathrm{Ric}(X, Y)$ is the *Ricci-form* $\rho(X, Y) \in \mathcal{A}^{1,1}_{\mathbb{R}}(V, J)^{orb}$ given by $\rho(X, Y) = \mathrm{Ric}(JX, Y)$. The proof of [HUY05, Proposition 4.A.11] generalizes in a straightforward manner to Kähler orbifolds, so the Ricci-form ρ satisfies $\rho(X, Y) = i\,\mathrm{tr}_{\mathbb{C}}(F_\nabla)$. Hence, $\rho \in 2\pi c_1(J)^{orb}$. A real $(1,1)$-form $\alpha \in \mathcal{A}^{1,1}_{\mathbb{R}}(V, J)^{orb}$ is positive, vanishing or negative if the corresponding real symmetric J-invariant form a is positive definite, vanishing or negative definite respectively. A class in $H^2_{dR}(V, \mathbb{R})$ is positive, vanishing or negative if it can be represented by a positive, vanishing or negative 2-form respectively. Calabi [CAL54] conjectured in 1954 that to any closed real $(1,1)$-form ρ in the first Chern class of a compact Kähler manifold (M, J, g) there exists a unique Kähler metric with ρ as its Ricci-form and whose Kähler form is cohomologous to the Kähler form of g. We provide a generalization of this result to compact Kähler orbifolds. *The Calabi conjecture for orbifolds* is

Theorem 3.4.3 (Calabi Conjecture, orbifolds)**.** *Let (V, J, g) be a compact Kähler orbifold with Kähler form ω. Then for any closed real $(1, 1)$-form ρ with $[\rho] = 2\pi c_1(J)^{orb}$ there exists a unique Kähler metric g' such that its Kähler form ω' satisfies $[\omega'] = [\omega] \in H^2(V, \mathbb{R})$ and ρ' is the Ricci-form of g'.*

The claim that the proof of Theorem 3.4.3 goes through largely unchanged on orbifolds has been made both in [JOY00, Theorem 6.5.6] and in [BOY-GAL08, Theorem 5.2.2]. While this might be clear to Joyce and Boyer-Galicki, we feel that it is nevertheless worthwhile to actually give a proof. We have included a proof of Theorem 3.4.3 in Section 3.5. We follow the proof of the Calabi conjecture for manifolds outlined in [JOY00, Chapter 5]. The continuity method works, as predicted by Joyce, and the theorem can be proved by locally viewing the orbifold as a quotient of a manifold by a finite group Γ and locally lifting the problem to a Γ-invariant problem on a manifold. We check that all local constructions preserve the Γ-invariance and glue to global constructions.

Deformation theory of complex structures

Deformation theory of complex structures on complex orbifolds can be defined similarly to the deformation theory for complex structures on manifolds we introduced in Section 2.6. For our purpose, the parameter space of deformations will always be smooth, so we can make do with a simplified smooth version of the deformation theory.

Definition 3.4.4. *Let $\mathcal{X}$ be a complex orbifold and S a smooth manifold with a proper holomorphic map $\pi : \mathcal{X} \to S$. The fibres $\mathcal{X}_t = \pi^{-1}(t)$ are compact complex suborbifolds of $\mathcal{X}$ and we say that $\pi : \mathcal{X} \to S$ is a* smooth family of complex orbifolds *parametrized by S.*

If we fix $0 \in S$ and restrict the family $\pi : \mathcal{X} \to S$ from Definition 3.4.4 to a germ around 0, then it can be viewed as a smooth family of deformations of the compact complex orbifold $V = \mathcal{X}_0$. This family of deformations can be trivialized as a differentiable family $\mathcal{X} \simeq S \times V$. In this way $\mathcal{X}_t$ is diffeomorphic to $\mathcal{X}_0$ for all t and $\pi : \mathcal{X} \to S$ can instead be viewed as a family of deformations of the complex structure J on (V, J). Kodaira and Spencer proved in [KOD-SPE60, Theorem 15] that for a compact Kähler manifold (M, J), small deformations are Kähler. This was generalized to orbifolds by El Kacimi Alaoui in [KAC88].

Theorem 3.4.5 (El Kacimi Alaoui)**.** *Let (V, J, g) be a compact Kähler orbifold and assume that there exists a smooth family of deformations $\pi : \mathcal{X} \to S$ of V. Then it admits a smooth family of compatible Kähler metrics g_t.*

Next we introduce infinitesimal complex deformations. A complex structure J satisfies the two equations $J^2 = -Id$ and $N(J) = 0$, where $N(J)$ is the Nijenhuis tensor, so the linearization $I = \frac{d}{dt} J_t|_{t=0} \in C^\infty(TV \otimes T^*V)^{orb}$ satisfies the two equations $0 = \frac{d}{dt}(-Id)|_{t=0} = \frac{d}{dt} J_t^2|_{t=0} = IJ + JI$ and $0 = N'_J(I) = \frac{1}{2} J \circ \bar{\partial} I$. For $X \in T^{1,0}V$ we have $JIX = -IJX = -iIX$ so $IX \in T^{0,1}V$ and $I \in \mathcal{A}^{0,1}(T^{1,0}V)^{orb}$. But the tensor I also satisfies $\bar{\partial} I = 0$, so it is an element of $H^1(V, \mathcal{T}_V)$. Tensor fields $I \in C^\infty(TX \otimes T^*X)$ satisfying the two equations $IJ + JI = 0$ and $N'_J(I) = 0$ and which are not of the form $L_X J$ are called *(essential) infinitesimal complex deformations* and we denote the space of such deformations by $ICD(J)^{orb}$. We say that an infinitesimal complex deformation I is *integrable* if there exists a smooth curve of deformation J_t with $J_0 = J$ for which $I = \frac{d}{dt} J_t|_{t=0}$. An orbifold V is said to have *unobstructed deformations* if all infinitesimal complex deformations integrate into a smooth curve of deformations.

Proposition 3.4.6. *Let (V, J) be a compact complex orbifold. Then there is a natural bijection between the space of infinitesimal complex deformations $ICD(J)^{orb}$ and $H^1(V, \mathcal{T}_V)$. If $H^2(V, \mathcal{T}_V) = 0$ then V has unobstructed deformations.*

Proof. adaptation to orbifolds of [KOD-NIR-SPE58, Theorem, p. 452]. □

For a compact orbifold (V, J) and a smooth family of complex deformations J_t, the orbifold first Chern class $c_1(J_t)^{orb}$ is stable for small deformations. This is because the denominator m_α for each branch divisor is stable under small deformations, so $c_1(J_t)^{orb}$, despite taking values in the image of $H^2(V, \mathbb{Q})$ in $H^2(V, \mathbb{R})$, actually only make integer value jumps.

Definition 3.4.7. *A* Calabi-Yau *orbifold is a compact Kähler orbifold with trivial canonical bundle.*

A Calabi-Yau orbifold V has structure group in $SU(m)$ and vanishing orbifold first Chern class. It also satisfies $H^2(V, \mathcal{T}_V) = 0$, so all infinitesimal complex deformations are integrable. This follows by adapting [HUY05, Prpoisition 6.1.11] to orbifolds.

3.5 Calabi conjecture

Calabi [CAL54] conjectured in 1954 that for any closed real $(1,1)$-form ρ' in $2\pi c_1(J)$ of a compact Kähler manifold (M, J, g) there exists a unique Kähler metric g' with ρ' as its Ricci-form and whose Kähler form is cohomologous to the Kähler form of g. The uniqueness part of the theorem was proved by Calabi himself. The existence part of the theorem was proved later by Yau [YAU78]. This is a fundamental existence result about Kähler metrics with prescribed Ricci-forms on Kähler manifolds. In particular the result tells us that on any Kähler manifold with vanishing first Chern class there exists a Ricci-flat Kähler metric. Joyce claimed in [JOY00, section 6.5] that the proof can be generalized to compact Kähler orbifolds by viewing an orbifold locally as a quotient of a manifold by a finite group Γ. Following this approach we adapt the proof of the Calabi conjecture for compact Kähler manifolds presented in [JOY00, chapter 6] to compact Kähler orbifolds using the orbifold first Chern class $c_1(J)^{orb}$. The Calabi Conjecture for orbifolds was stated in Section 3.1, but we restate it here for the convenient of the reader. We prove

Theorem 3.5.1. *Let (V, J, g) be a compact Kähler orbifold with Kähler form ω. Then for any closed real $(1,1)$-form ρ' with $[\rho'] = 2\pi c_1(J)^{orb}$ there exists a unique Kähler metric g' such that its Kähler form ω' satisfies $[\omega'] = [\omega] \in H^2(V, \mathbb{R})$ and ρ' is the Ricci-form of g'.*

The fist step of the proof is to reformulate Theorem 3.5.1 as an existence result for solutions to a complex Monge-Ampére equation. The reformulation hinges on the $\partial\bar{\partial}$-Lemma (Lemma 3.4.1) and ω^m being a volume form on V. Both hold on orbifolds and we get the following reformulation of Theorem 3.5.1.

Theorem 3.5.2. *Let (V, J, g) be a compact Kähler orbifold with Kähler form ω. Then for any $f \in C^{3,\alpha}(V)^{orb}$ there exists $A > 0$ and $\phi \in C^{5,\alpha}(V)^{orb}$ satisfying the following two conditions*

i) $\int_V \phi \, dV_g^{orb} = 0$,

ii) $(\omega + dd^c\phi)^m = Ae^f\omega^m$.

In a local holomorphic coordinate chart $\{U, \Gamma, \phi\}$ with coordinates $\{z^1, \ldots, z^m\}$ the equation $(\omega + dd^c\phi)^m = Ae^f\omega^m$ becomes

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}\right) = Ae^f \det(g_{\alpha\bar{\beta}}). \tag{3.2}$$

This equation is a *complex Monge-Ampére equation*. It is a non-linear partial differential equation of second order in ϕ. Note that it follows from part *ii)* that $\omega + dd^c\phi$ is a positive definite real $(1,1)$-form. For the rest of this section let (V, J, g) be a compact complex Kähler orbifold of complex dimension m and let ω be the Kähler form of g. We use the notation $C = C(X, \ldots, Z)$ to express that a constant C only depends on the parameters $X, \ldots, Z$.

Definition 3.5.3. *Fix $\alpha \in (0,1)$ and $f \in C^{3,\alpha}(V)^{orb}$. Define S to be the set of those $t \in [0,1]$ for which there exists a $\phi \in C^{5,\alpha}(V)^{orb}$ with $\int_V \phi \, dV_g^{orb} = 0$ and an $A > 0$ satisfying the equation $(\omega + dd^c\phi)^m = Ae^{tf}\omega^m$.*

The proof of Theorem 3.5.2 is based on the continuity method. The idea is to show that $1 \in S$. The space S is non-empty as for $t = 0$ the function $0 \in C^{5,\alpha}(V)^{orb}$ is a solution to (3.2). We show that $S = [0,1]$ by showing that it is both open and closed in $[0,1]$.

To show that S is open we use the inverse mapping theorem for Banach spaces. Let $X \subseteq C^{5,\alpha}(V)^{orb}$ be the vector space consisting of those ϕ for which $\int_V \phi \, dV_g^{orb} = 0$ and let $U = \{\phi \in X \,|\, \omega + dd^c\phi \text{ is a positive } (1,1)\text{-form}\}$. Suppose $\phi \in U$ and $a \in \mathbb{R}$, then there exists a unique real function f on V such that $(\omega + dd^c\phi)^m = e^{a+f}\omega^m$ on V, and as $\phi \in C^{5,\alpha}(V)^{orb}$, then $f \in C^{3,\alpha}(V)^{orb}$. Define a function

$$F : U \times \mathbb{R} \to C^{3,\alpha}(V)^{orb} : (\phi, a) \mapsto f,$$

where $(\omega + dd^c\phi)^m = e^{a+f}\omega^m$. The expression of the map F in local holomorphic coordinates $\{z^1, \ldots, z^m\}$ is

$$F(\phi, a) = \log\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}\right) - \log\det(g_{\alpha\bar{\beta}}) - a = f.$$

The expression of the Laplacian on a Kähler orbifold is the same as on a Kähler manifold, i.e. $\Delta\phi = -g^{\alpha\bar{\beta}}\partial_\alpha\bar{\partial}_{\bar{\beta}}\phi$. The Jacobi formula also holds on orbifolds. Denote by g_ϕ the metric $g_{\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}$. The linearization of F in (ϕ, a) is the map $DF_{(\phi,a)} : X \times \mathbb{R} \to C^{3,\alpha}(V)^{orb}$ given by

$$\begin{aligned} DF_{(\phi,a)}(u,b) &= L_{(\phi,a)} F(\phi + tu, a + tb) \\ &= \frac{d}{dt} F(\phi + tu, a + tb)|_{t=0} \\ &= \frac{d}{dt} \log\det(g_{\phi+tu})|_{t=0} - \frac{d}{dt}(a - tb)|_{t=0} \\ &= \frac{1}{\det(g_{\phi+tu})} \frac{d}{dt} \det(g_{\phi+tu})|_{t=0} - b \\ &= \frac{1}{\det(g_{\phi+tu})} \det(g_{\phi+tu}) \text{tr}((g_{\phi+tu})^{-1} \frac{d}{dt} g_{\phi+tu})|_{t=0} - b \\ &= \text{tr}((g_{\phi+tu})^{-1} \frac{d}{dt} \partial_\alpha \bar{\partial}_{\bar{\beta}}(tu))|_{t=0} - b \\ &= \text{tr}((g_\phi)^{-1} \partial_\alpha \bar{\partial}_{\bar{\beta}} u) - b \\ &= -\Delta_{g_\phi} u - b. \end{aligned} \tag{3.3}$$

To show that $(u, b) \mapsto -\Delta_{g_\phi} u - b$ is surjective onto $C^{3,\alpha}(V)^{orb}$ take $v \in C^{3,\alpha}(V)^{orb}$. The operator Δ_{g_ϕ} is elliptic, so solutions in $C^{5,\alpha}(V)^{orb}$ to $\Delta_{g_\phi} u - b = v$ exist by the orbifold version of Theorem 2.1.8 if $(v + b) \perp \ker(\Delta^*_{g_\phi})$. The Laplace operator is self-adjoint, so $\Delta^*_{g_\phi}(v) = e^f \Delta_{g_\phi}(e^{-f} v)$, and so $\ker(\Delta^*_{g_\phi})$ is the constant multiples of e^{-f}. We can always choose $b \in \mathbb{R}$ such that $(v+b) \perp e^{-f}$. A solution (u, b) to $-\Delta_{g_\phi} u - b = v$ therefore always exists. $DF|_{(\phi,a)}$ is therefore an invertible linear map. It is furthermore a homeomorphism, so the inverse function theorem for Banach spaces (Theorem 2.1.9) applies in the same way as in the proof of [JOY00, Theorem C3]. This allows us to conclude.

Theorem 3.5.4. *Fix $\alpha \in (0, 1)$ and suppose that $f' \in C^{3,\alpha}(V)^{orb}$ and $\phi' \in C^{5,\alpha}(V)^{orb}$ and $A' > 0$ satisfy the equations*

$$\int_V \phi' \, dV_g^{orb} = 0 \quad \textit{and} \quad (\omega + dd^c \phi')^m = A' e^{f'} \omega^m.$$

Then for every $f \in C^{3,\alpha}(V)^{orb}$ with $||f - f'||_{C^{3,\alpha}}$ sufficiently small, there exist $\phi \in C^{5,\alpha}(V)^{orb}$ and $A > 0$ such that

$$\int_V \phi \, dV_g^{orb} = 0 \quad \textit{and} \quad (\omega + dd^c \phi)^m = A e^f \omega^m. \tag{3.4}$$

Corollary 3.5.5. *The set S is open in $[0, 1]$.*

Proof. Let $t_0 \in S$, i.e. $\exists \phi \in C^{5,\alpha}(V)^{orb}$ and $A > 0$ such that $(\omega + dd^c\phi)^m = Ae^{t_0 f}\omega^m$. If we take $t \in [0, 1]$ sufficiently close to t_0, then $||tf - t_0 f||_{C^{3,\alpha}} = |t - t_0|\,||f||_{C^{3,\alpha}}$ is arbitrarily small, so by Theorem 3.5.4 there exist $\phi \in C^{5,\alpha}(V)^{orb}$ and $A > 0$ such that $(\omega + dd^c\phi)^m = Ae^{tf}\omega^m$. Hence $t \in S$. The set S is therefore open. □

To show that S is closed we prove that the limit point of any convergent sequence in S is in S. Yau used some hard a priori third order norm estimates to show this. The orbifold version of this is summarized in the next theorem.

Theorem 3.5.6. *Let $Q_1 \geq 0$. Then there exist $Q_2, Q_3, Q_4 \geq 0$ depending only on V, J, g and Q_1 such that the following holds: Suppose $f \in C^3(V)^{orb}, \phi \in C^5(V)^{orb}$ and $A > 0$ satisfy the equations*

$$||f||_{C^3} \leq Q_1, \quad \int_V \phi\, dV_g^{orb} = 0, \quad \text{and} \quad (\omega - dd^c\phi)^m = Ae^f\omega^m.$$

Then $||\phi||_{C^0} \leq Q_2, ||dd^c\phi||_{C^0} \leq Q_3$ and $||\nabla dd^c\phi||_{C^0} \leq Q_4$.

Proof. To prove the zero'th order estimate the idea is to find an L^k-bound on ϕ for each k and use this to bound the C^0-norm of ϕ. The Sobolev embedding theorem (Theorem 3.3.1), Stokes theorem and Hölder's inequality are used to find constants $C_4, Q_2 > 0$ depending only on V, g and Q_1 such that $||\phi||_{L^k} \leq Q_2(C_4 k)^{-m/k}$ for all $k \geq 2$. The theorems involved generalize to orbifolds and so does the properties of the constants C_4 and Q_2. The function ϕ is continuous and V is compact, so $||\phi||_{C^0} = \lim_{k\to\infty} ||\phi||_{L^k}$. Hence $||\phi||_{C^0}$ is bounded by Q_2 as $\lim_{k\to\infty} ||\phi||_{L^k} \leq \lim_{k\to\infty} Q_2(C_4 k)^{-m/k} = Q_2$.
The second order estimate $||dd^c\phi||_{C^0} \leq Q_3$ is based on a pointwise bound $|dd^c\phi|_g^2 \leq 2m + 2(m - \Delta\phi)^2$ depending only on m and $||\Delta\phi||_{C^0}$ and a C^0-bound on $\Delta\phi$ depending only on m, Q_1, Q_2 and g. Both bounds use a number of local inequalities involving $\nabla\phi$, $\Delta\phi$ and Riemann curvature tensor R, which all have identical expressions on orbifolds and manifolds. The estimates in [JOY00, section 5.4] therefore generalize to orbifolds.
The third order estimate $||\nabla dd^c\phi||_{C^0} \leq Q_4$ for a positive constant Q_4 depending only on V, J, g and Q_1 is a lengthy calculation, but uses only similar methods to the previous estimates and therefore also generalizes to orbifolds. The C^0-bound of $\Delta\phi$ is used to produce a C^0-bound on a non-negative function $S \in C^\infty(V)^{orb}$ given by $4S^2 = |\nabla dd^c|_{g'}^2$.

From the relation $|\nabla dd^c\phi|^2_{g'} \leq C^{3/2}|\nabla dd^c\phi|^2_{g'}$, where C is a constant depending only on V, J, g and Q_1, and the C^0-bound on S, we get the bound Q_4 on $||\nabla dd^c||_{C^0}$. □

Theorem 3.5.7. *Let $Q_1, Q_2, Q_3, Q_4 \geq 0$ and $\alpha \in (0,1)$. Then there exists $Q_5 \geq 0$ depending only on $V, J, g, Q_1, \ldots, Q_4$ and α, such that the following holds. Suppose $f \in C^{3,\alpha}(V)^{orb}, \phi \in C^5(V)^{orb}$ and $A > 0$ satisfy $(\omega + dd^c\phi)^m = Ae^f\omega^m$ and the inequalities*

$$||f||_{C^{3,\alpha}} \leq Q_1, \quad ||\phi||_{C^0} \leq Q_2, \quad ||dd^c\phi||_{C^0} \leq Q_3 \quad \text{and} \quad ||\nabla dd^c\phi||_{C^0} \leq Q_4.$$

Then $\phi \in C^{5,\alpha}(V)^{orb}$ and $||\phi||_{C^{5,\alpha}} \leq Q_5$. Also, if $f \in C^{k,\alpha}(V)^{orb}$ for some $k \geq 3$ then $\phi \in C^{k+2,\alpha}(V)^{orb}$, and if $f \in C^\infty(V)^{orb}$ then $\phi \in C^\infty(V)^{orb}$.

The proof of Theorem 3.5.7 is based on the inductive process known as bootstrapping. The first part of the proof is to generalize three regularity results with Schauder type norm bounds to the orbifold setting. The first lemma is an application of Theorem 3.3.4.

Lemma 3.5.8. *Let $k \geq 0$ and $\alpha \in (0,1)$. Then there exists a positive constant $E_1 = E_1(k, \alpha, V, g)$ such that if $\phi \in C^2(V)^{orb}$ satisfies $\Delta\phi = f$ for some $f \in C^{k,\alpha}(V)^{orb}$, then $\phi \in C^{k+2,\alpha}(V)^{orb}$ and*

$$||\phi||_{C^{k+2,\alpha}} \leq E_1 \left(||\Delta\phi||_{C^{k,\alpha}} + ||\phi||_{C^0}\right).$$

Lemma 3.5.9. *Let $\alpha \in (0,1)$. Then there exists a constant $E_2 = E_2(\alpha, V, g, ||g'_{ab}||_{C^0}, ||g'^{ab}||_{C^{0,\alpha}}) > 0$ such that if $\phi \in C^2(V)^{orb}$ satisfies $\Delta'\phi = f$ for some $f \in C^0(V)^{orb}$, then $\phi \in C^{1,\alpha}(V)^{orb}$ and*

$$||\phi||_{C^{1,\alpha}} \leq E_2 \left(||\Delta'\phi||_{C^0} + ||\phi||_{C^0}\right).$$

Proof. Let $f \in C^0(V)^{orb}$ and assume we have a solution $\phi \in C^2(V)^{orb}$ to $\Delta'\phi = f$. V is compact, so we can choose two finite collections of charts $\{B_1(0), \Gamma_{1,i}, \phi_{1,i}\}$ and $\{B_2(0), \Gamma_{2,i}, \phi_{2,i}\}$ covering V such that for each i they satisfy $\phi_{1,i}(B_1(0)) \subset \phi_{2,i}(B_2(0))$. Remember that $C^k(B_2(0))_\Gamma \subset C^k(B_2(0))$ for all k, so the solution to $\Delta'\phi = f$ locally satisfies $\phi \in C^2(B_2(0))$. For each i Theorem 1.4.3 of [JOY00] tells us that $\phi|_{B_1(0)} \in C^{1,\alpha}(B_1(0))$ and that there exists $D_i = D_i(m, \alpha) > 0$ such that the $C^{1,\alpha}$-norm of $\phi|_{B_1(0)}$ satisfies $||\phi|_{B_1(0)}||_{C^{1,\alpha}} < D_i(||\Delta'\phi||_{C^0} + ||\phi||_{C^0})$. But $\phi|_{B_1(0)}$ is Γ-invariant, and therefore

$\phi|_{B_1(0)} \in C^{1,\alpha}(B_1(0))_\Gamma$. Setting $E_2 = \sup_i\{D_i\}$ gives us the desired constant for the Schauder estimate for Δ' on V. □

Lemma 3.5.10. *Let* $k \geq 0$ *and* $\alpha \in (0,1)$*, then there exists a constant* $E_3 = E_3(k, V, \alpha, g, ||g'_{ab}||_{C^0}, ||g'^{ab}||_{C^{0,\alpha}}) > 0$ *such that if* $\phi \in C^2(V)^{orb}$ *satisfies* $\Delta'\phi = f$ *for* $f \in C^{k,\alpha}(V)^{orb}$*, then* $\phi \in C^{k+2,\alpha}(V)^{orb}$ *and*

$$||\phi||_{C^{k+2,\alpha}} \leq E_3\,(||\Delta'\phi||_{C^{k,\alpha}} + ||\phi||_{C^0}).$$

Proof. The proof is similar to that of Lemma 3.5.9 except that we use equation (1.14) instead of (1.13) from Theorem 1.4.3 of [JOY00]. □

Proof of Theorem 3.5.7. This is proved via the inductive process known as bootstrapping. It only uses already established results on orbifolds and norm bound estimates similar to what we have already seen, so it generalizes to orbifolds. We summarize the argument. In the proof of Theorem 3.5.6 we established a bound on $||\Delta\phi||_{C^0}$. From Lemma 3.5.8 and Lemma 3.5.9 it then follows that $\phi \in C^{3,\alpha}(V)^{orb}$ and that $||\phi||_{C^{3,\alpha}}$ is bounded by a constant depending only on $V, g, J, Q_1, \ldots, Q_4$ and α. Lemma 3.5.8 and Lemma 3.5.10 then show that if $f \in C^{k,\alpha}(V)^{orb}$ then $\phi \in C^{k+2,\alpha}(V)^{orb}$. An inductive argument together with Lemma 3.5.8 applied to $\Delta\phi$ shows that $\Delta\phi \in C^{k,\alpha}(V)^{orb}$. From this we get an a priori bound on $||\phi||_{C^{k+2,\alpha}}$ depending only on $V, g, J, Q_1, \ldots, Q_4$ and α. As $f \in C^{k,\alpha}(V)^{orb}$ implies $\phi \in C^{k+2,\alpha}(V)^{orb}$, f smooth implies ϕ smooth. □

Corollary 3.5.11. *The set* S *is closed in* $[0,1]$.

Proof. Consider a sequence $\{t_j\}_{j=0}^{\infty} \subset S$ with limit $t' \in [0,1]$. By definition there are sequences $\{\phi_j\}_{j=0}^{\infty}$ and $A_j > 0$ satisfying $i)$ and $ii)$ of Theorem 3.5.2. By Theorem 3.5.6 and 3.5.7 the sequence $\{\phi_j\}_{j=0}^{\infty}$ is bounded in $C^{5,\alpha}(V)^{orb}$. By Kondraschov's Theorem for orbifolds (Theorem 3.3.2) there exists a convergent subsequence $\{\phi_{j_k}\}_{k=0}^{\infty}$ with limit $\phi' \in C^5(V)^{orb}$ which satisfies $i)$ and $ii)$ of Theorem 3.5.2. From Theorem 3.5.6, and 3.5.7 it follows that the limit is indeed in $C^{5,\alpha}(V)^{orb}$. The set S therefore contains its limit points and it is therefore closed in $[0,1]$. □

This concludes the proof of Theorem 3.5.2. It remains to be shown that the metric found in Theorem 3.5.1 is indeed unique.

Theorem 3.5.12. *Let (V, J) be a compact complex orbifold and g a Kähler metric with Kähler form ω. Let $f \in C^1(V)^{orb}$. Then there is at most one function $\phi \in C^3(V)^{orb}$ satisfying $\int_V \phi \, dV_g^{orb} = 0$ and $(\omega + dd^c\phi)^m = Ae^f\omega^m$ on V for $A > 0$.*

Proof. This is an application of Stokes' theorem with a straight forward generalization to orbifolds and we therefore omit the proof. See [JOY00, Theorem C4] for details. □

3.6 Slice construction

In this section, we construct a slice for the action of the diffeomorphism group in the space of metrics on a compact orbifold. This is a generalization of a similar slice construction for compact manifolds made by Ebin [EBI70]. A *slice* around a point is a subset containing the point such that each orbit passing through the subset has exactly one representative in it. Slices are usually constructed by exponentiating the orthogonal complement of the tangent space of the orbits. We call the equation that has this orthogonal complement as its kernel the *slice equation*.

We will follow the same line of arguments as Ebin but work equivariantly on the orbifold charts and carefully check that all constructions carry over to orbifolds. The application of the slice construction we have in mind does not require a specific slice, which allows us to simplify the argument by Ebin considerably by avoiding a number of technical arguments about smoothness of particular bundles.

The fact that Proposition 3.3.3 holds on orbifolds as well as manifolds is not a profound statement, but is nevertheless a central ingredient that makes the generalization of Ebin's construction from compact manifolds to compact orbifolds work.

Let (V, g) be a compact orbifold and denote by $\mathcal{M}^{orb}$ the space of smooth (orbifold) Riemannian metrics on V. It is an open subset of $C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$. In Section 3.3 we gave $C^k(\mathrm{Sym}^2(T^*V))^{orb}$ the topology of uniform convergence and it is a Banach space in this norm. The space of smooth sections is the Fréchet space $C^\infty(\mathrm{Sym}^2(T^*V))^{orb} = \cap_{k=0}^\infty C^k(\mathrm{Sym}^2(T^*V))^{orb}$. Denote by $(\cdot,\cdot)_g^{\otimes i}$ the inner product on $\mathrm{Sym}^2(T^*V) \otimes (T^*V)^{\otimes i}$ induced by the metric g. The space $L_k^2(\mathrm{Sym}^2(T^*V))^{orb}$ is a Hilbert space with respect to the inner product $(\eta, \gamma)_{L_k^2} = \sum_{i=0}^k \int_V (\nabla^i \eta_p, \nabla^i \gamma_p)_g^{\otimes i} dV_g^{orb}$. Denote by $C^k\mathcal{M}^{orb}$ the space of C^k (orbifold) Riemannian metrics in $C^k(\text{Sym}^2(T^*V))^{orb}$ and define

$$\mathcal{M}_k^{orb} = L_k^2(\text{Sym}^2(T^*V))^{orb} \cap C^0\mathcal{M}^{orb}.$$

Lemma 3.6.1. *Let (V, g) be a compact orbifold of dimension n and let $k > n/2$. Then $\mathcal{M}_k^{orb}$ is a Banach manifold inside $L_k^2(\text{Sym}^2(T^*V))^{orb}$.*

Proof. The space $C^0(\text{Sym}^2(T^*V))^{orb}$ is a Banach space by Proposition 3.3.3. The space $C^0\mathcal{M}^{orb}$ is open in $C^0(\text{Sym}^2(T^*V))^{orb}$. Theorem 3.3.1 tells us that for $k > n/2$ the map $L_k^2(\text{Sym}^2(T^*V))^{orb} \to C^0(\text{Sym}^2(T^*V))^{orb}$ is continuous, so $L_k^2(\text{Sym}^2(T^*V))^{orb}$ is open in $C^0(\text{Sym}^2(T^*V))^{orb}$. The space $\mathcal{M}_s^{orb}$ is therefore the intersection of two open sets in $C^0(\text{Sym}^2(T^*V))^{orb}$. □

Consider the product orbifold $F = V \times V$ as a trivial fibre bundle over V with orbifold fibres V. C^k sections of this bundle are exactly the C^k orbifold maps from V to V. We equip the space of C^k-sections of this bundle, $C^k(F)^{orb}$, with the topology of uniform convergence. The space of C^k-diffeomorphisms from V to V can alternatively be described as

$$C^k\mathcal{D}^{orb} = \{f \in C^k(F)^{orb} \mid f \text{ has an inverse and } f^{-1} \in C^k(F)^{orb}\}.$$

Define

$$\mathcal{D}_k^{orb} = L_k^2(F)^{orb} \cap C^1\mathcal{D}^{orb}.$$

Lemma 3.6.2. *For $k > n/2 + 1$ the space $\mathcal{D}_k^{orb}$ is a Hilbert manifold.*

Proof. Theorem 3.3.1 applies to sections of the bundle $F = V \times V$. The inclusion $L_k^2(F)^{orb} \to C^1(F)^{orb}$ is therefore continuous and $L_k^2(F)^{orb} \cap C^1(F)^{orb}$ is an open subset in $L_k^2(F)^{orb}$. Proposition 3.3.3 applies to $L_k^2(F)^{orb}$ as well, so $\mathcal{D}_k^{orb}$ is an open subset in a Hilbert space. □

The group $\mathcal{D}_{k+1}^{orb}$ acts on $\mathcal{M}_k^{orb}$ by pull-back. We denote this action by

$$A : \mathcal{D}_{k+1}^{orb} \times \mathcal{M}_k^{orb} \to \mathcal{M}_k^{orb} : (\eta, g) = \eta^* g.$$

The space of (smooth) diffeomorphisms $\mathcal{D}^{orb}$ acts on $\mathcal{M}^{orb}$ in the same way. We also denote that action by A. For (V, g) compact and $k \geq \frac{n}{2}$, the Banach manifold $\mathcal{M}_k^{orb}$ has tangent space $L_k^2(\mathrm{Sym}^2(T^*V))^{orb}$. The inner product on $\mathcal{M}_k^{orb}$ that we get from the L_k^2-inner product $(\eta, \gamma)_{L_k^2}^g$ is invariant under the $\mathcal{D}_{k+1}^{orb}$-action. The argument is identical to the manifold case. See [EBI70, p.20-21] for details. We denote by I_g the *isometry group* $\{\eta \in \mathcal{D}^{orb} \,|\, \eta^* g = g\}$ of the metric $g \in \mathcal{M}^{orb}$. Before proving Theorem 3.1.2 we start by producing a slice for the action of $\mathcal{D}_{k+1}^{orb}$ on $\mathcal{M}_k^{orb}$ for $k > n/2 + 2$.

Theorem 3.6.3. *Let V be a compact orbifold and let $k > n/2 + 2$. For each $g \in \mathcal{M}^{orb}$ there exists a submanifold $S = S_g \subseteq \mathcal{M}_k^{orb}$ containing g for which*

i) If $\eta \in I_g$ then $A(\eta, S) = S$.

ii) If $\eta \in \mathcal{D}_{k+1}^{orb}$ satisfies $A(\eta, S) \cap S \neq \emptyset$ then $\eta \in I_g$.

iii) There exists a neighbourhood $U \subseteq \mathcal{D}_{k+1}^{orb}/I_g$ around the identity coset and a local section $\chi : U \to \mathcal{D}_{k+1}^{orb}$ such that

$$F : U \times S \to \mathcal{M}_k^{orb} : (\eta, g) \mapsto A(\chi(\eta), g)$$

is a homeomorphism onto a neighbourhood of g in $\mathcal{M}_k^{orb}$.

The construction of the cross-section χ is identical to the one for manifolds. We refer the reader to [EBI70, Proposition 5.10] for details.

Lemma 3.6.4. *The projection map $\pi : \mathcal{D}_{k+1}^{orb} \to \mathcal{D}_{k+1}^{orb}/I_g$ admits a smooth local cross section at any $\eta \in \mathcal{D}_{k+1}^{orb}/I_g$.*

Denote by O_g^k the orbit $(\mathcal{D}_{k+1}^{orb})^* g = (\mathcal{D}_{k+1}^{orb}/I_g)^* g$. Denote by ν the normal bundle to $T(O_g^k)$ in $T(\mathcal{M}_k^{orb})|_{O_g^k}$ with respect to the restricted metric from $\mathcal{M}_k^{orb}$ and let $\exp$ be the exponential map on the manifold $\mathcal{M}_k^{orb}$ with respect to the inner product given above. We are now ready to prove Theorem 3.6.3.

Proof of Theorem 3.6.3. let $U' \subset O_g^k$ be an open subset such that it has a section $\chi : U' \to \mathcal{D}_{k+1}^{orb}$. For $\epsilon' > 0$ let $V' = \{v \in \nu_g \,|\, ||v||_g \leq (\epsilon')^2\}$ and $W' = \{d(\eta^*)v \,|\, v \in V' \text{ and } \eta \in \chi(U')\}$. We can choose $\epsilon' > 0$ such that $\exp|_{W'}$ is a diffeomorphism onto its image and $\exp(W') \cap O_g^k = U'$. For some $\delta > 0$ we have $B_{2\delta}(g) \subset \exp(W')$. Now choose $U \subset U'$

and $\epsilon < \epsilon'$ and let V and W be defined as before but with ϵ and U in place of ϵ' and U'. Let ϵ and U be chosen such that $\exp(W) \subset B_\delta(g)$. Define $S = \exp(V)$.

Claim 3.6.5. *The set S has the three properties of a slice.*

For the first property, let $\eta \in I_g$ and $x \in S$ with $x = \exp(v)$ for some $v \in V$. Then η satisfies $\eta^* g = g$ so η is also an isometry for the inner product on $\mathcal{M}_k^{orb}$. Hence $||\eta^* v||_g = ||v||_g$ so $\eta^* v \in V$. For any isometry $\phi \in I_g$ the exponential map satisfies $\phi \circ \exp_p(v) = \exp_{\phi(p)}(d\phi_p(v))$ and η^* is linear so $d\eta^* = \eta^*$. Hence,

$$\eta^* \exp_g(v) = \exp_{\eta^* g}(d\eta^*(v)) = \exp_{\eta^* g}(\eta^* v).$$

It follows that $\eta^* x = \eta^* \exp(v) = \exp(\eta^* v) \in S$.

For the second property, let $\eta \in \mathcal{D}_{k+1}^{orb}$ and assume we have $x, y \in S$ with $\eta^*(x) = y$. Denote by $d(x, y)$ the distance function on $\mathcal{M}_k^{orb}$ given by the metric on $\mathcal{M}_k^{orb}$ introduced above. We have to show that $\eta \in I_g$. First $\eta^* x \in S$ and $S \subset W$ so $d(\eta^* x, g) < \delta$. η^* is an isometry, so $d(\eta^* x, \eta^* g) = d(x, g) < \delta$. This gives us,

$$d(g, \eta^* g) \leq d(g, y) + d(y, \eta^* g) = d(g, y) + d(\eta^* x, \eta^* g) = \delta + d(x, g) < 2\delta.$$

So $\eta^* g \in \exp(W')$. The exponential map $\exp|_{W'}$ is a diffeomorphism onto a neighbourhood of g in $\mathcal{M}_k^{orb}$ and satisfies that for $x, y \in S$ with $y = \eta^* x$ both $a, b \in W$ and $\exp(b) = y = \eta^* x = \eta^* \exp(a) = \exp(\eta^* a)$. By injectivity of $\exp$ we have $b = \eta^* a$, and as $a, b \in V'$, the definition of W' tells us that $\eta \in I_g$ as η does not move a out of the V'.

For the third property, let $U \subseteq O_g^k$, $\chi : U \to \mathcal{D}_{k+1}^{orb}$ and W be given as above. As $\exp|_W$ is a diffeomorphism onto a neighbourhood of g, then $F : U \times S \to \mathcal{M}_k^{orb} : (u, x) \mapsto A(\chi(u), x)$ is a bijection onto $\exp(W)$ and is continuous. Now, let $z \in \exp(W)$ and let $\pi : \nu \to O_g^k$ be the bundle projection map, then we can express the inverse of F as follows

$$F^{-1}(z) = (\, \pi \circ \exp^{-1}(z)\, ,\, A((\chi \circ \pi \circ \exp^{-1}(z))^{-1}, z)\,). \tag{3.5}$$

This map is continuous so F is a homeomorphism onto a neighbourhood of g in $\mathcal{M}_k^{orb}$. This concludes the proof of Claim 3.6.5. □

For the convenience of the reader we restate Theorem 3.1.2 here. It is a smooth version of Theorem 3.6.3.

Theorem 3.6.6. *For each $g \in \mathcal{M}^{orb}$ there exists a submanifold $S = S_g \subseteq \mathcal{M}^{orb}$ containing g for which*

i) If $\eta \in I_g$ then $A(\eta, S) = S$.

ii) If $\eta \in \mathcal{D}^{orb}$ satisfies $A(\eta, S) \cap S \neq \emptyset$ then $\eta \in I_g$.

iii) There exists a neighbourhood $U \subseteq \mathcal{D}^{orb}/I_g$ around the identity coset and a local section $\chi : U \to \mathcal{D}^{orb}$ such that

$$F : U \times S \to \mathcal{M}^{orb} : (\eta, g) \mapsto A(\chi(\eta), g)$$

is a homeomorphism onto a neighbourhood of $g \in \mathcal{M}^{orb}$.

Proof. The sets S and U from Theorem 3.6.3 we rename S_k and U_k respectively and use S and U to denote $S = S_k \cap \mathcal{M}^{orb}$ and $U = U_k \cap \mathcal{D}^{orb}$. We show that S has the three properties of a slice.

For the first property, let $\eta \in I_g$. The metric $\eta^* g'$ is smooth for each $g' \in S$. In particular $A(\eta, S) = S$.

For the second property, any $\eta \in \mathcal{D}^{orb}_{k+1}$ satisfies the second property of a slice by Theorem 3.6.3. In particular, any $\eta \in \mathcal{D}^{orb}$ for which $A(\eta, S) \cap S \neq \emptyset$ satisfies $\eta \in I_g$.

For the third property, the map χ defined in Theorem 3.6.3 maps U to $\mathcal{D}^{orb}$ and gives a local cross section $\chi : U \to \mathcal{D}^{orb} : aI_g \mapsto a$. Denote by ν the normal bundle to the tangent bundle of the orbit $(\mathcal{D}^{orb}/I_g)^* g$ inside $T(\mathcal{M}^{orb})$. Denote by $V \subset \nu_g$ the subset satisfying $\exp(V) = S$. Let $W = \{d(\eta^*)y \,|\, y \in V, \eta \in \chi(U)\}$. The exponential map is a local diffeomorphism and the map F satisfies $F(U \times S) = \exp(W) \cap \mathcal{M}^{orb}$, so it is continuous and bijective onto a neighbourhood of g in $\mathcal{M}^{orb}$. From Equation (3.5) it follows that F^{-1} is continuous. The map F is therefore a homeomorphism onto its image. This concludes the proof. □

For a compact orbifold (V, g) the set S constructed above provides a slice in $\mathcal{M}^{orb}$ through g for the action of $\mathcal{D}^{orb}/I_g$, i.e. S is homeomorphic to a neighbourhood of $[g]$ in $\mathcal{M}^{orb}/(\mathcal{D}^{orb}/I_g)$. The map $S \times \mathcal{D}/I_g \to \mathcal{M}^{orb} : (s, \eta I_g) \mapsto \chi(\eta I_g)^* s$ constructed in Theorem 3.6.6 induces a homeomorphism $S/I_g \to \mathcal{M}^{orb}/\mathcal{D}^{orb}$ onto a neighbourhood of

$[g]$. This construction restricts to the volume 1-metrics and produces a homeomorphism $(S \cap \mathcal{M}_1^{orb})/I_g \to \mathcal{M}_1^{orb}/\mathcal{D}^{orb}$ onto a neighbourhood of $[g]$. So we have,

Corollary 3.6.7. *Let (V, g) be a compact orbifold, then S/I_g is homeomorphic to a neighbourhood of the structure $[g]$ in $\mathcal{M}^{orb}/\mathcal{D}^{orb}$. This construction restricts to a slice $(S \cap \mathcal{M}_1^{orb})/I_g$ for the action of $\mathcal{D}^{orb}$ on $\mathcal{M}_1^{orb}$.*

3.7 Ricci-flat deformations

In this section, we study Ricci-flat deformations and infinitesimal Ricci-flat deformations of Ricci-flat metrics. At the end of this section we give a proof of Theorem 3.1.3, which is the first of the two main results of Chapter 3.

Let (V, g) be a compact orbifold. A smooth curve of *deformations* of a metric g is a smooth curve $g : (-\epsilon, \epsilon) \to C^\infty(\mathrm{Sym}^2(T^*V))^{orb} : t \mapsto g_t$ with $g_0 = g$. Any such family of deformations can be written as $g_t = g + h_t$ for a smooth curve $h_t \in C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$ with $h_0 = 0$. To keep notation simple we denote both the individual deformations and the smooth curve of deformations by g_t. Positive definiteness is an open condition, so for a deformation g_t with $|h_t|_g < 1$ small enough, the deformation g_t is positive definite. In the following we will often be discussing *small deformations* g_t, i.e. deformations g_t for which $\epsilon > 0$ is sufficiently small, or equivalently that $|h_t|_g$ is sufficiently small for the given context.

A smooth curve of Einstein deformations of an Einstein metric g is a smooth curve of deformations g_t such that g_t is Einstein for each t. The Einstein operator E introduced in Definition 2.6 generalizes to a map

$$E : \mathcal{M}_1^{orb} \to C^\infty(\mathrm{Sym}^2(T^*V))^{orb} : g \mapsto \mathrm{Ric}(g) - \frac{T_g}{n} g,$$

where $T_g = \int_V s_g \, dV_g^{orb}$ is the total scalar curvature of g. An orbifold metric g is Einstein exactly when $E(g) = 0$. One difference between the operator E on an orbifold and on a manifold is the order of isotropy appearing as a denominator in the total scalar curvature on orbifolds as explained in Section 3.2. We will be working with Ricci-flat metrics, so this difference will not play a role for our purpose. The operators $d, \nabla_g, \delta_g, \delta_g^*, \Delta_L$ and $\mathring{R}$ as defined in Section 2.3 generalize naturally to orbifolds following the constructions in

Section 3.2. We will often omit the subscript g from these operators when there is no risk of confusion.

From now on assume that the compact orbifold (V, g) is Ricci-flat. The operators involved have the same expressions on manifolds and orbifolds and the expression for the linearized Einstein operator is the same on manifolds and orbifolds. Let g_t be a smooth curve of deformations of g and let $h \in C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$ be the first jet $h = \frac{d}{dt} g_t|_{t=0} = \frac{d}{dt}(g + h_t)|_{t=0} = \frac{d}{dt} h_t|_{t=0}$. From Equation 2.8 it follows that the linearization of $E(g_t)$ is

$$\begin{aligned}
\frac{d}{dt} E(g_t)|_{t=0} &= \mathrm{Ric}'_g(h) \\
&= \frac{1}{2}\Delta_L(h) - \delta_g^* \delta_g h - \frac{1}{2}\nabla_g d(\mathrm{tr}_g h) \qquad (3.6)\\
&= \left(\frac{1}{2}\nabla^*\nabla - \overset{\circ}{R}\right) h - \delta^*\delta h - \frac{1}{2}\nabla d(\mathrm{tr}_g h). \qquad (3.7)
\end{aligned}$$

Assume that the metric g has volume 1. By Theorem 3.6.6 there exists a slice $S_g \subset \mathcal{M}_1^{orb}$ through g. Denote by $P = P_g$ the set of all Ricci-flat metrics in $S = S_g$. It is called the *premoduli space* of Ricci-flat metrics. We equip it with the subspace topology.

Definition 3.7.1. *Let (V, g) be a compact Ricci-flat orbifold. Define the space $\epsilon(g)^{orb} \subset C^\infty(Sym^2(T^*V))^{orb}$ by*

$$\epsilon(g)^{orb} = \{h \in C^\infty(Sym^2(T^*V))^{orb} \,|\, h \text{ satisfies } i), ii) \text{ and } iii)\}$$

where

$$i)\ Ric'_g(h) = 0 \qquad ii)\ \delta h = 0 \qquad iii)\ \int_V tr_g h = 0. \qquad (3.8)$$

The first jet of a smooth curve of Ricci-flat deformations g_t of g belongs to the space $\epsilon(g)^{orb}$. Without further assumptions the space $\epsilon(g)^{orb}$ might contain elements that are not first jets of Ricci-flat deformations of g. Later we will impose sufficient conditions on V and g such that every element of $\epsilon(g)^{orb}$ is indeed a first jet of a Ricci-flat deformation of g. With this in mind we name $\epsilon(g)^{orb}$ the *space of infinitesimal Ricci-flat deformations*.

Lemma 3.7.2. *Let (V, g) be a compact orbifold. A tensor field $h \in C^\infty(Sym^2(T^*V))^{orb}$*

belongs to $\epsilon(g)^{orb}$ if and only if it satisfies

$$(\nabla^*\nabla - 2\overset{\circ}{R})h = 0 \qquad \delta h = 0 \qquad tr_g h = 0. \tag{3.9}$$

Proof. (3.9) $\Rightarrow$ (3.8): It is immediate that the equations in (3.9) imply conditions $i), ii)$ and $iii)$ of Definition 3.7.1.

(3.8) $\Rightarrow$ (3.9): Denote by $Hess(f)$ the $(2,0)$-tensor $\nabla_g d(f)$. To see that $i), ii)$ and $iii)$ of Definition 3.7.1 imply equations (3.9) first observe that $\delta h = 0$ and $\text{Ric}'_g(h) = 0$ simplifies equation (3.6) to

$$0 = \text{Ric}'_g(h) = \frac{1}{2}(\Delta_L h - Hess(\text{tr}_g h)).$$

Remember that $\text{tr}_g(\Delta_L h) = \Delta(\text{tr}_g h)$ and $\text{tr}_g(Hess(f)) = -\Delta f$. If we take trace on both sides of the equation $\Delta_L h = Hess(\text{tr}_g h)$, then [BER-EBI69, p. 388-389] gives us

$$\Delta(\text{tr}_g h) = \text{tr}_g(Hess(\text{tr}_g h)) = -\Delta(\text{tr}_g h).$$

The function $\text{tr}_g h$ is therefore harmonic. The only harmonic functions on V, however, are the constant functions, so $\int_V \text{tr}_g h = 0$ implies $\text{tr}_g h = 0$. From Equation (3.7) it follows that $(\nabla^*\nabla - 2\overset{\circ}{R})h = 0$. □

Lemma 3.7.3. *Let (V, g) be a compact Ricci-flat orbifold. The space $\epsilon(g)^{orb}$ is finite dimensional.*

Proof. The space $\epsilon(g)^{orb}$ is in the kernel of the elliptic operator $\nabla^*\nabla - 2\overset{\circ}{R}$ and it is therefore finite dimensional by Theorem 3.3.5. □

Theorem 3.7.4. *Let (V, g) be a compact Ricci-flat orbifold. Then there exists a finite dimensional real analytic submanifold $Z \subset S_g$ with $T_g Z = \epsilon(g)^{orb}$ and with P_g as a real analytic subset.*

Proof. Fix $k > n + 2$ and let $S_g^k \subset \mathcal{M}_k^{orb}$ be the slice generated by Theorem 3.6.3 and let S_1^k be the subset of volume 1 metrics. The elliptic operator $F = 2\text{Ric}'_g + 2\delta^*\delta = \Delta_L - \nabla d\text{tr}_g : L_k^2(\text{Sym}^2(T^*V))^{orb} \to L_{k-2}^2(\text{Sym}^2(T^*V))^{orb}$ is Fredholm, so it has closed image. By the same argument as in [BES87, Lemma 12.48] $F(T_g S_1^k)$ is closed as well. The operator Ric satisfies $\text{Ric}'_g(T_g S_1^k) = F(T_g S_1^k)$ and it is real analytic according to

[DeT-KAZ81]. The space $L^2_k(\mathrm{Sym}^2(T^*V))^{orb}$ is a Hilbert space by Proposition 3.3.3 and we can therefore apply Theorem 2.1.11 to conclude that there exists a neighbourhood U of g in S^k_1 such that $U \cap P_g$ is a real analytic subset of a real analytic submanifold $Z \subset U$ whose tangent space $T_g Z$ coincides with $\ker(\mathrm{Ric}'_g) \cap T_g(S^k) = \epsilon^2_k(g)^{orb}$. This is true for every $k > n+2$. In particular, it is true for the slice S_1. By Lemma 3.7.3 the space $\epsilon(g)^{orb}$ is finite dimensional. □

For the rest of this section, let (V, J) be a compact complex orbifold of complex dimension m. For $h \in C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$ define the tensor

$$h \circ J(X, Y) = h(X, JY).$$

A tensor field $h \in C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$ is said to be *Hermitian* if $h(JX, JY) = h(X, Y)$ and *skew-Hermitian* if $h(JX, JY) = -h(X, Y)$. Any $h \in C^\infty(\mathrm{Sym}^2(T^*V))^{orb}$ can be written as $h = h_A + h_H$ with h_H Hermitian and h_A skew-Hermitian.

For a smooth curve of complex deformations J_t, $I = \frac{d}{dt} J_t|_{t=0} \in C^\infty(TX \otimes T^*X)$ satisfies $-Id = J_t^2$ and $N(J_t) = 0$, where $N(J_t)$ is the Nijenhuis tensor of J_t. Differentiating $-Id = J_t^2$ gives us $0 = IJ + JI$, and differentiating $N(J_t) = 0$ gives $0 = N'_J(I) = \frac{1}{2} J \circ \bar{\partial} I$. The tensor $g \circ I$ satisfies $g \circ I(JX, JY) = g(JX, IJY) = -g(JX, JIY) = -g \circ I(X, Y)$, so it is skew-Hermitian. We say that $I \in ICD(J)^{orb}$ is *symmetric* or *anti-symmetric* if $g \circ I$ is symmetric or anti-symmetric respectively. Denote by I^s and I^a the symmetric part and skew-symmetric part of I respectively. Both I^a and I^s belong to $ICD(J)^{orb}$. Denote by $ICD(J)^{orb}_S$ and $ICD(J)^{orb}_A$ the subspace of symmetric and skew-symmetric infinitesimal complex deformations respectively.

For a *Hermitian* symmetric tensor h_H, the tensor $\psi = h_H \circ J$ satisfies $\psi(X, Y) = h_H(X, JY) = h_H(JX, J^2Y) = -h_H(Y, JX) = -\psi(Y, X)$, so it is a real differential 2-form. Also, $\mathbf{J}\psi(X, Y) = \psi(JX, JY) = h_H(JX, J^2Y) = -h(JX, Y) = -h(J^2X, JY)$ $= h(X, JY) = i^{1-1}\psi(X, Y)$, so ψ is of type $(1, 1)$. Hence $\psi = h_H \circ J \in \mathcal{A}^{1,1}_{\mathbb{R}}(V, J)^{orb}$.

It will be convenient to rewrite the equations from (3.9) for a Hermitian tensor and a skew-Hermitian tensor respectively. We start with the Hermitian case. Let $h_H \in C^\infty(\mathrm{Sym}^2(T^*V))$ be a Hermitian tensor and let $\psi = h_H \circ J$. The description of Kähler metrics from Remark 2.5.2 holds also on orbifolds, so the metric g is Kähler with respect to the complex structure J exactly if $\nabla J = 0$, or equivalently if $J \nabla_Y X = \nabla_Y JX$ for all $X, Y \in T_p V$ and for all $p \in V$. Letting $\{Y_i\}_{i=1}^{2m}$ be an orthonormal frame and using this relation gives us

$$
\begin{aligned}
-(d^*\psi)\circ J(X) &= -(\nabla^*\psi)(JX) = -\sum_{i=1}^{2m} \nabla_{Y_i}\psi(Y_i, JX)\\
&= -\sum_{i=1}^{2m} Y_i(\psi(Y_i, JX)) - \psi(\nabla_{Y_i}Y_i, JX) - \psi(Y_i, \nabla_{Y_i}JX)\\
&= -\sum_{i=1}^{2m} Y_i(h_H(Y_i, J^2X)) - h_H(\nabla_{Y_i}Y_i, J^2X) - h_H(Y_i, J\nabla_{Y_i}JX)\\
&= \sum_{i=1}^{2m} Y_i(h_H(Y_i, X)) - h_H(\nabla_{Y_i}Y_i, X) - h_H(Y_i, \nabla_{Y_i}X)\\
&= \nabla^*h_H(X) = \delta_g h_H(X)
\end{aligned}
$$

with the conclusion

$$\delta_g h_H = -(d^*\psi)\circ J. \tag{3.10}$$

Let ω be the Kähler form of g with respect to J. As for manifolds, a local calculation shows that

$$\mathrm{tr}_g h_H = (\psi, \omega)_g. \tag{3.11}$$

Similarly to [BES87, (12.92')], there is a *Weitzenböck formula* for Hermitian tensors on the Ricci-flat orbifold (V, g)

$$\Delta\psi = (\nabla^*\nabla - 2\overset{\circ}{R})h_H \circ J. \tag{3.12}$$

A skew-Hermitian symmetric tensor h_A can be identified with an infinitesimal complex deformation in the following way. Let $I \in End(TV)^{orb}$ be the real endomorphism satisfying

$$h_A \circ J = g \circ I. \tag{3.13}$$

This endomorphism I satisfies

$$\begin{aligned} g(X, IJY) &= h_A(X, J^2Y) = -h_A(X,Y) = h_A(JX, JY) = g(JX, IY) \\ &= -g(JX, -IY) = -g(JX, J^2IY) = -g(X, JIY), \end{aligned} \tag{3.14}$$

which translates to $g(X,(IJ+JI)Y) = 0$ and I therefore anti-commutes with J. This implies, for $X \in T^{0,1}V$, that $J(IX) = -IJX = -I(-iX) = iIX$, so $IX \in T^{1,0}V$ and $I : T^{0,1}V \to T^{1,0}V$. I may therefore be regarded as an element of $\mathcal{A}^{0,1}(T^{1,0}V)^{orb}$. The tensor field $\bar{\partial}^* I \in C^\infty(T^{1,0}V)^{orb}$ is, by [BES87, (12.93)], related to $\delta_g h_A \in C^\infty(T^*V)^{orb}$ via

$$\delta_g h_A = -J \circ \bar{\partial}^* I. \tag{3.15}$$

In particular, $\delta_g h_A = 0$ exactly when I is $\bar{\partial}^*$-closed, i.e. when I determines a class in $H^1(V, \mathcal{T}_V)$. We need a version of the Weitzenböck formula for the complex Laplacian $\Delta_{\bar{\partial}}$. The Weitzenböck formula is essentially proved by expanding the definition of $\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} - \bar{\partial}\bar{\partial}^*$. The expression of $\bar{\partial}$ and $\bar{\partial}^*$ and the way they act on tensors is unchanged on orbifolds. The involved operators ∇, ∇^* and $\overset{\circ}{R}$ also have the same expressions on orbifolds as they had on manifold. The Weitzenböck decomposition on manifolds found in [BES87, equation (12.93')] therefore extends to orbifolds. We have

$$(\nabla^*\nabla - 2\overset{\circ}{R})h_A \circ J = g \circ \Delta_{\bar{\partial}} I. \tag{3.16}$$

Hence $(\nabla^*\nabla - 2\overset{\circ}{R})h_A = 0$ exactly when $I \in \mathcal{A}^{0,1}(T^{1,0}V)^{orb}$ is in the kernel of $\Delta_{\bar{\partial}}$.

Proposition 3.7.5. *Let (V, J, g) be a compact Ricci-flat Kähler orbifold and let $h \in \epsilon(g)^{orb}$ be decomposed as $h = h_H + h_A$. Then $h_H, h_A \in \epsilon(g)^{orb}$.*

Proof. We show that $h_A \in \epsilon(g)^{orb}$. It then follows that $h_H = h - h_A \in \epsilon(g)^{orb}$ as $\epsilon(g)^{orb}$ is a vector space. We have

$$0 = (\nabla^*\nabla - 2\overset{\circ}{R})h = (\nabla^*\nabla - 2\overset{\circ}{R})h_H + (\nabla^*\nabla - 2\overset{\circ}{R})h_A.$$

The operator $(\nabla^*\nabla - 2\overset{\circ}{R})$ preserves type so the skew-Hermitian tensor $(\nabla^*\nabla - 2\overset{\circ}{R})h_A$ must vanish. The Weitzenböck formula (3.16) gives us $\Delta_{\bar{\partial}} I = 0$, which in turn implies

$\bar{\partial} I = 0$. It then follows from (3.15) that $\delta_g h_A = 0$. The tensor h_A is skew-hermitian, so its trace is necessarily zero. The tensor h_A satisfies the three equations in (3.9), so $h_A \in \epsilon(g)^{orb}$. □

For the compact Kähler orbifold (V, J, g) let J_t be a smooth curve of deformations of J and let g_t be a corresponding smooth curve of deformations of g such that each g_t is Kähler with respect to J_t. Denote by h the tensor field $\frac{d}{dt} g_t|_{t=0}$ and by I the tensor field $\frac{d}{dt} J_t|_{t=0}$. Identify the $(1,1)$-tensor $I = I^i_j$ with the $(2,0)$-tensor $g \circ I$ and denote by I_{ij} the components of $g \circ I$. According to [KOI83, section 9] the tensor fields h and I are related via

$$2ih_{\alpha\beta} + (I_{\alpha\beta} + I_{\beta\alpha}) = 0, \tag{3.17}$$

$$2i(D_\alpha h_{\beta\bar{\gamma}} - D_\beta h_{\alpha\bar{\gamma}}) = D_{\bar{\gamma}}(I_{\alpha\beta} - I_{\beta\alpha}), \tag{3.18}$$

where we use Greek indices as explained in the preliminaries. We say that a symmetric $(2,0)$-tensor h and an infinitesimal complex deformation I satisfying (3.17) and (3.18) are *Kähler related.*

The space $\epsilon(g)^{orb}$ can be written as $\epsilon(g)^{orb} = \epsilon(g)^{orb}_H \oplus \epsilon(g)^{orb}_A$, where $\epsilon(g)^{orb}_H$ and $\epsilon(g)^{orb}_A$ denote the subspaces of $\epsilon(g)^{orb}$ of Hermitian and skew-Hermitian tensor fields respectively. In the following two subsections the subspaces $\epsilon(g)^{orb}_H$ and $\epsilon(g)^{orb}_A$ will be studied separately.

Hermitian symmetric 2-tensors

Let (V, J, g) be a compact Ricci-flat Kähler orbifold. We explained above that for a Hermitian symmetric $(2,0)$-tensor h, the tensor $h \circ J$ is a real differential 2-form of type $(1,1)$. For $h \in \epsilon(g)^{orb}_H$ the Weitzenböck formula (3.12) gives us

$$\Delta(h \circ J)(X, Y) = (\nabla^*\nabla - 2 \overset{\circ}{R})h(X, JY) = 0.$$

This way we have identified elements of $\epsilon(g)^{orb}_H$ with elements of $\mathcal{H}^{1,1}_{\mathbb{R}}(V, J)$. Denote by ω the Kähler form of g. We saw in (3.11) that $\text{tr}_g h = (h \circ J, \omega)_g$. The tensor h is trace-free, so the form $h \circ J$ is orthogonal to ω and therefore so is $[h \circ J]$ to $[\omega]$. We explained in Section 3.2 that for compact V we have $\mathcal{H}^{1,1}_{\mathbb{R}}(V, J) \simeq H^{1,1}_{\mathbb{R}}(V, J)$. The map

$\epsilon(g)^{orb}_H \to \mathcal{H}^{1,1}_{\mathbb{R}}(V,J)/\mathbb{R}\cdot\omega : h \mapsto h\circ J$ is linear. If $h_1 \circ J = h_2 \circ J$ then $h_1 = h_2$ so the map is injective. To see that the map is surjective take $\psi \in \mathcal{H}^{1,1}_{\mathbb{R}}(V,J)/\mathbb{R}\cdot\omega$ and define $h(X,Y) = \psi(JX,Y)$. Differential forms are skew-symmetric, so h is symmetric. The form ψ satisfies $\psi(JX,JY) = i^{1-1}\psi(X,Y)$, so h is Hermitian. By (3.11) we that $\mathrm{tr}_g h = (\psi,\omega)_g = 0$. The form ψ is harmonic, so (3.10) implies $\delta h = 0$. From (3.12) it follows that $(\nabla^*\nabla - 2\overset{\circ}{R})h = 0$. Hence, $h \in \epsilon(g)^{orb}_H$. The map is an isomorphism of vector spaces and we have shown the next proposition.

Proposition 3.7.6. *Let (V,J,g) be a compact Ricci-flat Kähler orbifold and let ω be the Kähler form of g. Then the map*

$$\begin{aligned} \epsilon(g)^{orb}_H &\longrightarrow \mathcal{H}^{1,1}_{\mathbb{R}}(V,J)/\mathbb{R}\cdot\omega \\ h &\mapsto h\circ J \end{aligned}$$

is an isomorphism and $\dim\epsilon(g)^{orb}_H = \dim H^{1,1}_{\mathbb{R}}(V,J) - 1.$

Lemma 3.7.7. *All $h \in \epsilon(g)^{orb}$ Kähler related to 0 are Hermitian.*

Proof. Take $h \in \epsilon(g)^{orb}$ Kähler related to 0. Equation 3.17 becomes $h_{\alpha\beta} = 0$ for all α,β. Expanding the expression $h_{\alpha\beta}$, we see that $h_{\alpha\beta} = -\frac{i}{4}((h\circ J)_{ab} + (h\circ J)_{ba})$. The tensor $h\circ J$ is therefore skew-symmetric. We obtain h from $\psi = h\circ J$ via $h(X,Y) = -\psi(X,JY)$. Hence,

$$h(JX,JY) = -\psi(JX,J^2Y) = \psi(J^2Y,JX) = h(J^2Y,J^2X) = h(X,Y),$$

so h is Hermitian. □

Skew-Hermitian symmetric 2-tensors

Let (V,J,g) be a compact Ricci-flat Kähler orbifold with volume 1. From the above remarks about anti-Hermitian tensors $h_A \in \epsilon(g)^{orb}_A$ we know that an endomorphism I satisfying $h_A \circ J = g\circ I$ defines a $\bar\partial$-closed element $I \in \mathcal{A}^{0,1}(T^{1,0}V)^{orb}$. Also,

$$\begin{aligned} g\circ I(X,Y) &= h_A\circ J(X,Y) = h_A(X,JY) = -h_A(JX,J^2Y) \\ &= h_A(JX,Y) = h_A(Y,JX) = h_A\circ J(Y,X) \\ &= g\circ I(Y,X) \end{aligned} \tag{3.19}$$

so $g \circ I$ is symmetric. Furthermore,

$$g \circ I(JX, JY) = g(JX, IJY) = -g(JX, JIY) = -g(X, IY) = -g \circ I(X, Y).$$

So the tensor $g \circ I$ is skew-Hermitian.

Proposition 3.7.8. *Let (V, J, g) be a compact Ricci-flat Kähler orbifold. Then the map*

$$\begin{aligned} \epsilon(g)^{orb}_A &\longrightarrow ICD(J)^{orb}_S \\ h &\longmapsto I \end{aligned}$$

where $h \circ J = g \circ I$, is an isomorphism onto the $\bar\partial$-harmonic elements.

Proof. We showed above that any $h \in \epsilon(g)^{orb}_A$ produces a symmetric and $\bar\partial$-harmonic infinitesimal complex deformation $I \in ICD(J)^{orb}_S$ via the correspondence $h \circ J = g \circ I$.

To see that $\epsilon(g)^{orb}_A$ surjects onto the space of harmonic elements in $ICD(J)^{orb}_S$, take I be a $\bar\partial$-harmonic element of $ICD(J)^{orb}_S$. The inverse map is $I \mapsto h$, where h is determined by $h(X, Y) = -g(JX, IY)$. It is indeed an inverse, as $h \circ J(X, Y) = h(X, JY) = -g(JX, IJY) = g(JX, JIY) = g \circ I(X, Y)$. The tensor h is well-defined on orbifolds, as both g and I are well-defined on orbifolds and the operations defining the inverse h are well-defined on orbifolds. We check that the inverse h belongs to $\epsilon(g)^{orb}_A$. The tensor h is symmetric as

$$h(X, Y) = -g(JX, IY) = -g \circ I(Y, JX) = -g \circ I(JY, J^2X) = -g(JY, IX),$$

and h is skew-Hermitian as

$$h(JX, JY) = -g(J^2X, IJY) = g(J^2X, JIY) = g(JX, IY) = -h(X, Y). \quad (3.20)$$

The tensor h is δ-closed by (3.15) and it is trace-free as it is skew-Hermitian. The tensor I is $\bar\partial$-harmonic, so $(\nabla^*\nabla - 2 \overset{\circ}{R})h = 0$ by the Weitzenböck formula (3.16). □

Corollary 3.7.9. *Let (V, J, g) be a compact Ricci-flat Kähler orbifold. Then*

$$\dim \epsilon(g)^{orb}_A = 2 \dim_{\mathbb{C}} H^1(V, \mathcal{T}_V) - 2 \dim_{\mathbb{C}} H^{0,2}(V, J).$$

Proof. First we show that $I \in ICD(J)^{orb}_A$ is harmonic exactly if $g \circ I$ is harmonic. By the Weitzenböck formula (3.16), the form $g \circ I$ satisfies $(\nabla^*\nabla - 2 \overset{\circ}{R})(g \circ I) = 0$ exactly if I is $\bar{\partial}$-harmonic. If $g \circ I$ is anti-symmetric then $\overset{\circ}{R} (g \circ I) = 0$, see [KOI83, Lemma 7.1] for details. Hence $\nabla^*\nabla(g \circ I) = 0$. This is equivalent to $g \circ I$ being parallel, i.e. $\nabla(g \circ I) = 0$. Being parallel implies being harmonic as $d(g \circ I)$ is the skew-symmetric part of $\nabla(g \circ I)$ and $g \circ I$ parallel implies $*g \circ I$ parallel. Hence $I \in ICD(J)^{orb}_A$ is harmonic exactly if $g \circ I$ is. That $I \in \mathcal{A}^{0,1}(T^{1,0}V)^{orb}$ is anti-symmetric means that $g \circ I$ is a differential 2-form and using g to identify $T^{1,0}V$ with $\wedge^{0,1}V$ it follows that $ICD(J)^{orb}_A \simeq \mathcal{H}^{0,2}(V, J)$. Following the construction in Section 3.4 the subspace of $\bar{\partial}$-harmonic forms in $ICD(J)^{orb}$ is isomorphic to $H^1(V, \mathcal{T}_V)$. □

Corollary 3.7.10. *Each $h \in \epsilon(g)^{orb}_A$ is Kähler related to some $g \circ I$ for an element $I \in ICD(J)^{orb}_S$ and this relation is equivalent to the relation $h \circ J = g \circ I$.*

Proof. Let $h \in \epsilon(g)^{orb}_A$. Then $h \circ J = g \circ I$ for an element $I \in ICD(J)^{orb}_S$ by Proposition 3.7.8. The relation between h and $g \circ I$ can alternatively be phrased as $h_{\alpha\beta} = -i(g \circ I)_{\alpha\beta}$ which, as $g \circ I$ is symmetric, is equivalent to Equation 3.17. Both sides of Equation 3.18 vanish. □

We are now ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. Let (V, J, g) be a compact complex orbifold with vanishing first Chern class. Assume that all infinitesimal complex deformations are integrable and let g be a Ricci-flat Kähler metric with respect to J.

Denote by $\mathcal{J}$ the parameter space of nearby complex structures on V and let $0 \in \mathcal{J}$ correspond to $J_0 = J$. It follows from the hypothesis that the parameter space $\mathcal{J}$ is smooth in a neighbourhood of 0 and it follows from Theorem 3.4.5 that for every $t \in \mathcal{J}$ the deformed complex structure J_t admits a compatible Kähler metric g_t. Consider the vector bundle $\mathcal{V}$ over $\mathcal{J}$ with fibre $\mathcal{V}_t$ at $t \in \mathcal{J}$ the space of real differential 2-forms which are harmonic with respect to g_t, of type $(1, 1)$ with respect to J_t and orthogonal to ω_t with respect to the metric induced on $\mathcal{H}^{1,1}_{\mathbb{R}}(V, J_t)$ by g_t. The sign of the first Chern class is stable for small deformations, so $c_1(J_t)^{orb}$ vanishes.

Take $h \in \epsilon(g)^{orb}_A$ and let $I \in ICD(J)^{orb}_S$ be Kähler related to h. All infinitesimal complex deformations are integrable, so there is a smooth curve of complex structure J_t with $I = \frac{d}{dt} J_t|_{t=0}$. Let g_t be a smooth curve of Kähler metrics with respect to J_t and denote by ω_t the Kähler form of g_t. Set $\kappa = (g_0' - h) \circ J$ and consider a smooth curve $(t, \kappa_t) \in \mathcal{V}$ of Hermitian tensors κ_t with $\kappa_0 = \kappa$. For sufficiently small t the tensor $\tilde{\omega}_t = \omega_t - t\kappa_t$ is a Hermitian positive definite closed real $(1,1)$-form with respect to J_t. Denote by $\tilde{g}_t$ the Kähler metric of $\tilde{\omega}_t$.

A solution $u \in C^5(V)^{orb}$ to the complex Monge-Ampére equation

$$(\omega_t - t\kappa_t + i\partial\bar{\partial}u)^n - Ae^f(\omega_t - t\kappa_t)^n = 0 \tag{3.21}$$

produces a C^3-real $(1,1)$-form in the class of $\omega_t - t\kappa_t$ which is Ricci-flat and Kähler with respect to J_t. We already have a solution $u = 0$ to this equation for $t = 0$. Let X be the vector subspace of $C^{5,\alpha}(V)^{orb}$ consisting of those u for which $\int_V u\, dV^{orb}_{\tilde{g}_t} = 0$. For each t the metric $\tilde{g}_t$ has Ricci-form $\tilde{\rho}_t \in 2\pi c_1(J_t) = 0$. We can therefore find a smooth curve $f_t \in C^\infty(V)^{orb}$ satisfying $\tilde{\rho}_t = i\partial\bar{\partial}f_t$. Following Section 3.5 we define a map $F : \mathcal{V} \times X \times \mathbb{R} \to C^{3,\alpha}(V)^{orb}$ by

$$F((t,\kappa_t), u, a) = \log\det\left((\tilde{g}_t)_{\alpha\bar{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}\right) - \log\det((\tilde{g}_t)_{\alpha\bar{\beta}}) - f_t - a.$$

For the sake of readability we abbreviate the metric $((\tilde{g}_t)_{\alpha\bar{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}})$ by $((\hat{g}_t)_{\alpha\bar{\beta}})$. In (3.3) we calculated

$$DF_{(t,\kappa_t,u,a)}|_{X\times\mathbb{R}}(h,b) = -\Delta_{\hat{g}}h - b,$$

and we showed that $DF_{(0,0)}|_{X\times\mathbb{R}}$ is a linear homeomorphism. So the implicit function theorem (Theorem 2.1.10) applies and there exists a smooth map $\psi : \mathcal{U} \to X \times \mathbb{R}$ from a neighbourhood $\mathcal{U}$ of $(0,0)$ in $\mathcal{V}$ to a neighbourhood of $(0,0)$ in $X \times \mathbb{R}$ such that for each $(J_t, \kappa_t) \in \mathcal{U}$ the metric $\hat{g}_t = \psi(J_t, \kappa_t)$ corresponding to the Kähler form $\hat{\omega}_t = \omega_t - t\kappa_t + i\partial\bar{\partial}u$ is Ricci-flat and Kähler with respect to the complex structure J_t. From [DeT-KAZ81, Theorem 6.1] it follows that C^2 Ricci-flat Kähler metrics are actually smooth, so ψ produces smooth metrics. We can restrict ψ to a subset $\tilde{\mathcal{U}}$ of $\mathcal{U}$ such that $\psi|_{\tilde{\mathcal{U}}}$ only takes values in S_g. Elements in the image of ψ are Ricci-flat, so $\psi(\tilde{\mathcal{U}})$ is contained in the premoduli space P_g. From Theorem 3.7.4 we know that $d\psi(T_{(0,0)}\tilde{\mathcal{U}}) \subset \epsilon(g)^{orb}$.

The rest of the proof consists in showing that ψ is a submersion onto P_g. Take the $h \in \epsilon(g)^{orb}_A$ from before which is Kähler related to $I = \frac{d}{dt}J_t|_{t=0}$. For the Kähler form

$\tilde{\omega}_t = \omega_t - t\kappa_t$, the form $\tilde{\omega}'_0 - h \circ J$ satisfies

$$\begin{aligned}
\tilde{\omega}'_0 - h \circ J &= \frac{d}{dt}(\tilde{g}_t \circ J_t)|_{t=0} - h \circ J \\
&= \frac{d}{dt}(\tilde{g}_t)|_{t=0} \circ J + \tilde{g} \circ I - h \circ J \\
&= \frac{d}{dt}(g_t - t\phi_t)|_{t=0} \circ J + \tilde{g} \circ I - h \circ J \\
&= (g'_0 - 1 \cdot \phi_0 - 0 \cdot \phi'_0) \circ J + \tilde{g} \circ I - h \circ J \\
&= \tilde{g} \circ I + (g'_0 - h) \circ J - \phi_0 \circ J \\
&= \tilde{g} \circ I + (g'_0 - h) \circ J - \kappa \\
&= \tilde{g} \circ I + 0 \\
&= \tilde{g} \circ I^a + \tilde{g} \circ I^s,
\end{aligned}$$

where we denoted by ϕ_t the tensor satisfying $\kappa_t = \phi_t \circ J_t$. The tensor $\tilde{g} \circ I^s$ is symmetric and the tensor $\tilde{g} \circ I^a$ satisfies

$$\begin{aligned}
\mathbf{J}(\tilde{g} \circ I^a)(X, Y) = \tilde{g} \circ I^a(JX, JY) &= \tilde{g}(JX, I^a JY) \\
&= -\tilde{g}(JX, JI^a Y) = -\tilde{g}(X, I^a Y) = -\tilde{g} \circ I^a(X, Y)
\end{aligned}$$

so the 2-form $\tilde{g} \circ I^a$ is not of type $(1,1)$. Hence $[\tilde{\omega}'_0 - h \circ J] = [\tilde{g} \circ I] = 0$ in $H^{1,1}(V, J)$. It is now clear why we added the form κ in (3.21). The class $[\tilde{\omega}'_0 - h \circ J]$ would otherwise not have been cohomologous to zero. Now,

$$\hat{\omega}'_0 = \frac{d}{dt}(\tilde{\omega}_t + i\partial\bar{\partial}u_t)|_{t=0} = \tilde{\omega}'_0 + i\partial\bar{\partial}u'_0,$$

so $[\hat{\omega}'_0] = [\tilde{\omega}'_0]$. Hence $[\hat{\omega}'_0 - h \circ J] = 0$.

The symmetric tensors $\hat{g}'_0$ and h are both Kähler related to $I = J'_0$, so $\hat{g}'_0 - h$ is Kähler related to 0. It is therefore Hermitian by Lemma 3.7.7. For the Hermitian tensor $\hat{g}'_0 - h \in \epsilon(g)^{orb}$ the form $(\hat{g}'_0 - h) \circ J$ is harmonic by the Weitzenböck formula (3.12) since

$$\Delta((\hat{g}'_0 - h) \circ J) = (\nabla^*\nabla - 2\overset{\circ}{R})(\hat{g}'_0 - h) \circ J = 0.$$

Now,

$$0 = [\hat{\omega}_0' - h \circ J] = [(\hat{g}_0' - h) \circ J + \hat{g} \circ I] = [(\hat{g}_0' - h) \circ J] + [\hat{g} \circ I]$$
$$= [(\hat{g}_0' - h) \circ J].$$

A harmonic form cohomologous to zero is zero, so $(\hat{g}_0' - h) \circ J = 0$, but this means that $\hat{g}_0' = h$. Hence, the differential of the map ψ is surjective and ψ is therefore a submersion. This concludes the proof. □

3.8 Moduli space of Ricci-flat structures

In this section, we study the moduli space of Ricci-flat structures. We prove that a neighbourhood of a Kähler structure is, up to an action of a finite group, a finite dimensional manifold and we find an expression for its dimension in terms of cohomology groups. This is the content of Theorem 3.1.4.

Let (V, J, g) be a compact complex orbifold. Introduce the following equivalence relation $\sim$ on $\mathcal{M}^{orb}$. Two Riemannian metrics g and g' are equivalent if for some $\phi \in \mathcal{D}^{orb}$ and some $c > 0$ they satisfy

$$g' = c\,\phi^* g.$$

Equivalence classes are called *Riemannian structures* and the quotient space $\mathcal{M}^{orb}/\sim$ is equipped with the quotient topology. We denote the quotient space by $\widetilde{\mathcal{M}}^{orb}$. Volume scales as $\text{vol}(cg)^{orb} = \sqrt{c^n}\,\text{vol}(g)^{orb}$, so the quotient space can equivalently be express as $\mathcal{M}_1^{orb}/\mathcal{D}^{orb}$. A *Ricci-flat structure* is a Riemannian structure containing a Ricci-flat metric. Denote by $\widetilde{\mathcal{M}}_{\mathcal{R}}^{orb}$ the subspace of Ricci-flat structures in $\widetilde{\mathcal{M}}^{orb}$. A diffeomorphism $\phi \in \mathcal{D}^{orb}$ is an isometry $\phi : (V, \phi^* g) \to (V, g)$ and it therefore satisfies $\text{Ric}(\phi^* g) = \phi^*\text{Ric}(g)$. From [BES87, Theorem 1.159] we know that

$$\text{Ric}(e^f g) = \text{Ric}(g) + (2-n)(\nabla df - df \otimes df) + (\Delta f - (n-2)||df||^2)g.$$

Rescaling by a constant c therefore satisfies $\text{Ric}(cg) = \text{Ric}(g)$. Denote by $\mathcal{R}^{orb}$ the subspace of $\mathcal{M}^{orb}$ of Ricci-flat structures. The space $\widetilde{\mathcal{M}}_{\mathcal{R}}^{orb}$ can therefore equivalently be expressed as $\mathcal{R}^{orb}/\sim$, which we therefore denote by $\widetilde{\mathcal{R}}^{orb}$. This space is called the *moduli space* of Ricci-flat structures.

For an orbifold (V, g) denote by $\mathrm{Iso}(V, g)^{orb}$ the group of isometries. The *identity component* of $\mathrm{Iso}(V, g)^{orb}$ is the connected component of the identity. We denote it by $(I_g^{orb})^0$. Myers and Steenrod showed in [MYE-STE39] that the isometry group of a Riemannian manifold is a Lie group and that if the manifold is compact, then so is the isometry group. This was generalized to orbifolds by Bagaev and Zhukova in [BAG-ZHU07, Corollary 1].

Theorem 3.8.1. *Let (V, g) be a compact orbifold. Then Iso$(V, g)^{orb}$ is a compact Lie group.*

Proposition 3.8.2. *Let (V, g) be a compact Ricci-flat orbifold. Then the identity component $(I_g^{orb})^0$ of the isometry group Iso$(V, g)^{orb}$ acts trivially on the premoduli space of Ricci-flat metrics P_g.*

Proof. From Theorem 3.1.2(2) it follows that any $\phi \in \mathcal{D}^{orb}$ satisfying $\phi^* S_g \cap S_g \neq \emptyset$ belongs to $\mathrm{Iso}(V, g)$. Hence $\mathrm{Iso}(V, g') \subset \mathrm{Iso}(V, g)$. In particular $(I_{g'}^{orb})^0 \subset (I_g^{orb})^0$. Similar to the case of compact Ricci-flat manifolds ([BES87, Theorem 1.84]), $\dim \mathrm{Iso}(V, g) = b_1(V, \mathbb{R}) = \dim \mathrm{Iso}(V, g')$, so the connected Lie groups $(I_{g'}^{orb})^0$ and $(I_g^{orb})^0$ coincide. Any $\eta \in (I_g^{orb})^0$ therefore acts trivially on P_g. □

Corollary 3.8.3. *Let (V, g) be a compact Ricci-flat orbifold. Then a neighbourhood of $[g]$ in $\widetilde{\mathcal{R}}^{orb}$ is homeomorphic to $P_g/(\mathrm{Iso}(V, g)^{orb}/(I_g^{orb})^0)$, where $\mathrm{Iso}(V, g)^{orb}/(I_g^{orb})^0$ is a finite group acting on the premoduli space P_g.*

Proof. The *identity component* $(I_g^{orb})^0$ is a normal subgroup of the compact Lie group $\mathrm{Iso}(V, g)^{orb}$. The quotient group $\mathrm{Iso}(V, g)^{orb}/(I_g^{orb})^0$, called the *group of components*, is therefore a finite group. A consequence of Theorem 3.1.2 and Proposition 3.8.2 is that a neighbourhood of $[g]$ in the moduli space $\widetilde{\mathcal{R}}^{orb}$ is locally homeomorphic to the quotient $P_g/\mathrm{Iso}(V, g)^{orb} = P_g/(\mathrm{Iso}(V, g)^{orb}/(I_g^{orb})^0)$. □

Let (V, J, g) be a compact complex orbifold. Define a *Kähler structure* on V to be a Riemannian structure containing a Kähler metric.

Proof of Theorem 3.1.4. Let (V, J, g) be a compact complex orbifold with vanishing first Chern class and all infinitesimal complex deformations integrable and let g be a Ricci-flat

Kähler metric on (V, J). By Theorem 3.1.2 there exists a slice $S_g \subset \mathcal{M}_1^{orb}$ for the action of $\mathcal{D}^{orb}$. From Theorem 3.7.4 we know that there exists a finite dimensional manifold $Z \subset S_g$ with $T_g Z = \epsilon(g)^{orb}$ and P_g as a real analytic subset. From Theorem 3.1.3 we know that all infinitesimal Ricci-flat deformations integrate into smooth curves of Ricci-flat deformations through g, so P_g spans an entire neighbourhood of g in Z and therefore satisfies $T_g P_g = \epsilon(g)^{orb}$. By Corollary 3.6.7 and Corollary 3.8.3 a neighbourhood $U \subset \widetilde{\mathcal{R}}^{orb}$ of $[g]$ is homeomorphic to $P_g/(\mathrm{Iso}(V, g)^{orb}/(I_g^{orb})^0)$.

To compute the dimension of $\widetilde{\mathcal{R}}^{orb}$ at $[g]$ we have to compute the dimension of P_g at g, which is $\dim \epsilon(g)^{orb}$. From Corollary 3.7.6 and Corollary 3.7.9 it follows that

$$\begin{aligned}\dim(\epsilon(g)^{orb}) &= \dim \epsilon(g)_H^{orb} + \dim \epsilon(g)_A^{orb} \\ &= \dim H_{\mathbb{R}}^{1,1}(V, J) - 1 + 2 \dim_{\mathbb{C}} H^1(V, \mathcal{T}_V) - 2 \dim_{\mathbb{C}} H^{0,2}(V, J).\end{aligned}$$

This concludes the proof. □

Chapter 4

Examples: Orbifold K3 surfaces

In this section, we provide examples of compact complex orbifolds with vanishing first Chern class and a Ricci-flat Kähler metric for which all infinitesimal complex deformations are integrable. We use Theorem 3.1.4 to calculate the dimension of the moduli space of Ricci-flat structures on these orbifolds. The examples are hypersurfaces in weighted projective spaces given by weighted homogeneous polynomials. For more details about these spaces see for instance [BOY-GAL08, Chapter 4], [JOY00, Chapter 6], [CAN-LYN90] and [DOL82].

Definition 4.0.4 (Weighted projective space)**.** *Let $m \geq 1$ be an integer and let $\mathbb{C}^{m+1}$ have coordinates $(z_0, \ldots, z_m)$. Let $\boldsymbol{a} = (a_0, \ldots, a_m)$ be a tuple of integers satisfying $a_0, \ldots, a_m \geq 1$ and $d_i = gcd(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) = 1$ for $i = 0, \ldots, m$. The weighted $\mathbb{C}^*$-action, denoted by $\mathbb{C}^*(\boldsymbol{a})$, on $\mathbb{C}^{m+1}\backslash\{0\}$ is defined by*

$$u.(z_0, \ldots, z_m) = (u^{a_0} z_0, \ldots, u^{a_m} z_m).$$

We define the weighted projective space $\mathbb{CP}^m_{a_0,\ldots,a_m}$ to be $(\mathbb{C}^{m+1}\backslash\{0\})/\mathbb{C}^*(\boldsymbol{a})$.

Note that if all weights are equal to 1 then a weighted projective space is the ordinary complex projective space. In the next example we construct an orbifold structure on $\mathbb{CP}^m_{a_0,\ldots,a_m}$ explicitly. The example is borrowed from [BOY-GAL08, p. 134].

Example 4.0.5. *Let $\mathbb{CP}^m_{a_0,\ldots,a_m}$ be the weighted projective space and fix an integer i with $0 \leq i \leq m$. We construct the local uniformizing system $\{U_i, \Gamma_i, \phi_i\}$ explicitly. Consider the set $U_i = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} \,|\, z_i = 0\}$ and let $\Gamma_i = \{z \in \mathbb{C}^* \,|\, z^{a_i} = 1\}$ be the*

a_i'th roots of unity. The finite group Γ_i acts on U_i via the weighted $\mathbb{C}^*(a_0,\ldots,a_m)$-action $\gamma.(z_0,\ldots,z_m) = (\gamma^{a_0}z_0,\ldots,\gamma^{a_m}z_m)$. Coordinates on U_i are $\boldsymbol{y} = (y_0,\ldots,\hat{y}_i,\ldots,y_m)$, where $\hat{y}_i$ means that y_i has been removed. We can cover $\mathbb{CP}^m_{a_0,\ldots,a_m}$ with coordinate patches of the form $\bar{U}_i = \{[z_0,\ldots,z_m] \in \mathbb{CP}^m_{a_0,\ldots,a_m} \mid z_i \neq 0\}$ with $0 \leq i \leq m$. Define the map $\phi_i : U_i \to \bar{U}_i : \boldsymbol{y} \mapsto \boldsymbol{y}^{a_i} = (y_0^{a_i},\ldots,\hat{y}_i^{a_i},\ldots,y_m^{a_i})$. It satisfies $\phi(\gamma.\boldsymbol{y}) = \phi(\boldsymbol{y})$ for all $\gamma \in \Gamma_i$, so it descends to a map $\phi_i : U_i/\Gamma_i \to \bar{U}_i$. The $\mathbb{C}^*(a_0,\ldots,a_m)$-orbits of the space $(\mathbb{C}^{m+1}\backslash\{0\})\backslash\{z_i = 0\}$ are in bijection with the points in U_i/Γ_i via the map ϕ_i. For each i the tuple $\{U_i,\Gamma_i,\phi_i\}$ is therefore a local uniformizing system. We can extend the family of l.u.s.'s for the open cover $\cup_{i=0}^m \bar{U}_i$ to an atlas by including charts of the form $\{U_{i_0}\cap\cdots\cap U_{i_k},\Gamma_{i_0\ldots i_k},\phi_{i_0\ldots i_k}\}$ where $\Gamma_{i_0\ldots i_k} = \mathbb{Z}_{gcd(a_{i_0}\ldots a_{i_k})}$ and the map $\phi_{i_0\ldots i_k} : U_{i_0}\cap\cdots\cap U_{i_k} \to \tilde{U}_{i_0}\cap\cdots\cap\tilde{U}_{i_k}$ is given by $\phi_{i_0\ldots i_k}(\boldsymbol{y}_{i_0\ldots i_k}) = \boldsymbol{y}_{i_0\ldots i_k}^{gcd(a_{i_0},\ldots,a_{i_k})}$. Injections can be created as follows. There is an open set $U_{ij} \subset U_i \cap U_j$ such that the injection map $\lambda_{ij,i} : U_{ij} \to U_i$ is $\lambda_{ij,i}(\boldsymbol{y}_{ij}) = \boldsymbol{y}_{ij}^t$ where $t = \frac{gcd(a_i,a_j)}{a_i}$. This satisfies the condition $\phi_i \circ \lambda_{ij,i} = \phi_{ij}$ and is well-defined with the action of Γ_i on U_i. We call this the standard orbifold structure on $\mathbb{CP}^m_{a_0,\ldots,a_m}$. We will always assume that $\mathbb{CP}^m_{a_0,\ldots,a_m}$ is equipped with this orbifold structure.

The next remark tells us how to find orbifold singularities in weighted projective spaces and what the isotropy groups at their singularities look like.

Remark 4.0.6. *Let $z = [z_0,\ldots,z_m] \in \mathbb{CP}^m_{a_0,\ldots,a_m}$ and let $\{z_{i_1},\ldots,z_{i_r}\}$ be a collection of those z_i's which equal zero. Let $\bar{U}_z = \{[z_1,\ldots,z_m] \in \mathbb{CP}^m_{a_0,\ldots,a_m} \mid z_{i_s} = 0 \, for \, s = 1,\ldots,r\}$ and let d be the greatest common divisor of those a_i for which $z_i \neq 0$. Let $\{U_z,\Gamma_z,\phi_z\}$ be the l.u.s. corresponding to $\bar{U}_z$. If $d = 1$ then $z = [z_0,\ldots,z_m]$ is a regular point. If $d > 1$ then $[z_0,\ldots,z_m]$ is a singular point with isotropy group $\mathbb{Z}_d$ and $\bar{U}_z$ is homeomorphic to $U_z/\mathbb{Z}_d$.*

Definition 4.0.7. *Let $f(z_0,\ldots,z_m) \in \mathbb{C}[z_0,\ldots,z_m]$ be a non-zero polynomial. We say that f is weighted homogeneous of degree d if*

$$f(u^{a_0}z_0,\ldots,u^{a_m}z_m) = u^d f(z_0,\ldots,z_m) \quad \forall u, z_0,\ldots,z_m \in \mathbb{C}.$$

Let $\mathbb{CP}^m_{a_0,\ldots,a_m}$ be a weighted projective space. We say that a subvariety $V \subseteq \mathbb{CP}^m_{a_0,\ldots,a_m}$ is *weighted homogeneous* if it is the common zero locus of a collection of weighted homogeneous polynomials. A subvariety $V \subset \mathbb{CP}^m_{a_0,\ldots,a_m}$ of a single weighted homogeneous polynomial f is called a (weighted homogeneous) *hypersurface* and we denote it by V_f. For a weighted homogeneous polynomial $f \in \mathbb{C}[z_0, \dots, z_m]$ we say that the hypersurface $V_f \subset \mathbb{CP}^m_{a_0,\dots,a_m}$ is *well-formed* if for all $0 \leq i < j \leq m$ the greatest common divisor $d_{ij} = gcd(a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_m)$ divides the degree of f. We say that V_f is *quasi-smooth of dimension* m if the partial derivatives of f do not vanish simultaneously. In the next lemma we summarize two results from [BOY-GAL08, p. 140-141].

Lemma 4.0.8. *Let $\mathbb{CP}^m_{a_0,\dots,a_m}$ and let $f(z_0, \dots, z_m) \in \mathbb{C}[z_0, \dots, z_m]$ be a weighted homogeneous polynomial and let $V_f \subseteq \mathbb{CP}^m_{a_0,\dots,a_m}$ be a quasi-smooth hypersurface defined by f, then the orbifold structure on $\mathbb{CP}^m_{a_0,\dots,a_m}$ induces on V_f the structure of an orbifold where all isotropy groups are cyclic. Furthermore, V_f is well-formed if and only if it has no branch divisors.*

The weighted projective space $\mathbb{CP}^m_{a_0,\dots,a_m}$ admits a generalization of the Fubini-Study metric. As explained in Example 4.0.5 the weighted projective space is covered by open sets of the form $\bar{U}_i = \{[z_0, \dots, z_m] \in \mathbb{CP}^m_{a_0,\dots,a_m} \mid z_i \neq 0\}$ with $0 \leq i \leq m$. The orbifold Fubini-Study metric is locally on $\bar{U}_i$ given by

$$\omega_i = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{m}\left|\frac{z_l^{a_i}}{z_i^{a_l}}\right|^2\right) \in \mathcal{A}^{1,1}(\bar{U}_i).$$

It patches to a global form and defined a Kähler metric on $\mathbb{CP}^m_{a_0,\dots,a_m}$. The following result from [JOY00, section 6.7] explains why well-formed and quasi-smooth hypersurfaces in weighted projective spaces provide applications of the main results of Chapter 3.

Theorem 4.0.9. *Let $\mathbb{CP}^m_{a_0,\dots,a_m}$ be a weighted projective space and let $f \in \mathbb{C}[z_0, \dots, z_m]$ be a weighted homogeneous polynomial of degree d for which the hypersurface V_f in $\mathbb{CP}^m_{a_0,\dots,a_m}$ defined by f is well-formed and quasi-smooth. Then V_f has trivial canonical bundle if and only if $d = a_0 + \dots + a_m$.*

The hypersurface V_f from Theorem 4.0.9 inherits a Kähler metric from the ambient weighed projective space $\mathbb{CP}^m_{a_0,\dots,a_m}$, so it is a Calabi-Yau orbifold. This implies that $H^2(V, \mathcal{T}_V) = 0$, so all infinitesimal complex deformations are integrable. Also, $\mathcal{T}_V \simeq \Omega^{m-1}_V$, so $H^1(V, \mathcal{T}_V) \simeq H^1(V, \Omega^{m-1}_V)$ and the dimension of $H^1(V, \mathcal{T}_V)$ is the Hodge number $h^{m-1,1}$. The arguments are similar to the smooth case. See e.g. [HUY05, section 6.1 and 6.2] for details. Next we give an example of an orbifold K3 surface

Example 4.0.10. *In the weighted projective space $\mathbb{CP}^3_{1,1,2,4}$ we consider the degree 8 weighted homogeneous polynomial $f(x,y,z,w) = x^8+y^8+z^4+w^2$. The greatest common divisors $d_{ij} \in \{1,2\}$ of the weights $(1,1,2,4)$ all divide 8, so f is well-formed. The polynomial f is known as a* Brieskorn-Pham polynomial. *Such polynomials are quasi-smooth. The hypersurface V_f is an orbifold of complex dimension 2 with isotropy $\mathbb{Z}_2$ singularities in $[0,0,z_1,z_2]$. The degree of f equals the sum of the weights $(1,1,2,4)$ so by Theorem 4.0.9 the hypersurface V_f is a Calabi-Yau orbifold. All infinitesimal complex deformations are also integrable on V_f, it has vanishing first Chern class and it admits a Ricci-flat Kähler metric. The orbifold V_f therefore satisfies the conditions of Theorem 3.1.3 and we can use Theorem 3.1.4 to compute the dimension of its moduli space of Ricci-flat structures in a neighbourhood of a Ricci-flat Kähler structure. It is explained in [BOY-GAL08, Appendix B.1] that the hypersurface V_f is an orbifold K3 surface with Hodge diamond*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 18 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

We have $\mathcal{T}_{V_f} \simeq \Omega^{2-1}_{V_f}$ so $H^1(V_f, \mathcal{T}_{V_f}) \simeq H^1(V_f, \Omega^1_{V_f})$ and so

$$\begin{aligned} \dim \epsilon(g)^{orb} &= h^{1,1}(V_f) - 1 + 2\,h^{1,1}(V_f) - 2\,h^{0,2}(V_f) \\ &= 18 - 1 + 2 \cdot 18 - 2 \\ &= 51. \end{aligned}$$

For orbifold K3 surfaces V the expression for the dimension of the moduli space of Ricci-flat structures from Theorem 3.1.4 simplifies to

$$\dim \epsilon(g)^{orb} = 3\,h^{1,1}(V) - 3 = 3\,b^2(V) - 9. \tag{4.1}$$

Boyer-Galicki reproduced in [BOY-GAL08, Appendix B.1] a list by Miles Reid from 1979 of 95 orbifold $K3$ surfaces arising as hypersurfaces in $\mathbb{CP}^3_{a_0,a_1,a_2,a_3}$ with varying Betti numbers. All orbifolds on the list satisfy the assumptions of Theorem 3.1.4. To compute the dimension of the moduli space of Ricci-flat structures for these orbifold K3 surfaces we simply need to input the relevant value for $h^{1,1}$ in the expression (4.1). We calculate the dimension of the moduli space of a few of the orbifold K3 surfaces listed in [BOY-GAL08, Appendix B.1].

Table 4.1: Orbifold K3 hypersurfaces

Ambient space	Degree	Polynomial	$\dim \epsilon(g)^{orb}$
$\mathbb{CP}^3_{1,1,4,6}$	12	$x^{12} + y^{12} + z^3 + w^2$	54
$\mathbb{CP}^3_{1,1,2,4}$	8	$x^8 + y^8 + z^4 + w^2$	51
$\mathbb{CP}^3_{1,1,2,2}$	6	$x^6 + y^6 + z^3 + w^3$	48
$\mathbb{CP}^3_{1,2,2,5}$	10	$x^{10} + y^5 + z^5 + w^2$	42
$\mathbb{CP}^3_{1,2,6,9}$	18	$x^{18} + y^9 + z^3 + w^2$	42
$\mathbb{CP}^3_{1,2,3,6}$	12	$x^{12} + y^6 + z^4 + w^2$	39
$\mathbb{CP}^3_{1,3,8,12}$	24	$x^{24} + y^8 + z^3 + w^2$	36
$\mathbb{CP}^3_{1,3,4,4}$	12	$x^{12} + y^4 + z^3 + w^3$	30
$\mathbb{CP}^3_{1,4,5,10}$	20	$x^{20} + y^5 + z^4 + w^2$	30
$\mathbb{CP}^3_{1,6,14,21}$	42	$x^{42} + y^7 + z^3 + w^2$	30
$\mathbb{CP}^3_{2,3,3,4}$	12	$x^6 + y^4 + z^4 + w^3$	24
$\mathbb{CP}^3_{2,3,10,15}$	30	$x^{15} + y^{10} + z^3 + w^2$	24

Chapter 5

ALE Ricci-flat deformations

5.1 Introduction and results

In this chapter, we extend results by Koiso from [KOI83] (see Section 2.7 for a review) about Ricci-flat deformations of Ricci-flat Kähler metrics on compact manifolds to a class of complete non-compact manifolds known as asymptotically locally Euclidean(ALE) manifolds. We show that ALE Ricci-flat deformations of ALE Ricci-flat Kähler metrics are Kähler, possibly with respect to a perturbed complex structure, and we show that the moduli space of ALE Ricci-flat structures is, up to the action of a finite group, a finite dimensional manifold and we find an expression for its dimension.

ALE metrics have the following model. Let $G \subset SO(n)$ be a finite subgroup acting freely on $\mathbb{R}^n\backslash\{0\}$ and let h_0 be the standard Euclidean metric on $\mathbb{R}^n$. h_0 is invariant under G so it descends to a metric on $(\mathbb{R}^n\backslash\{0\})/G$, which we also denote by h_0. An ALE manifold (X, g) is a non-compact Riemannian manifold with one end such that outside a compact set it is diffeomorphic to $(\mathbb{R}^n\backslash B_R(0))/G$ and such that the metric g approximates the flat metric h_0 towards infinity.

To define ALE metrics let ∇ be the Levi-Civita covariant derivative of h_0 and let $r : \mathbb{R}^n/G \to [0, \infty)$ be the radius function on $(\mathbb{R}^n\backslash\{0\})/G$ given by h_0. We use the big O-notation $f(x) = O(r^k)$ to express that there exist constants $R, C > 0$ such that $|f(x)| \leq Cr^k$ for all $|x| > R$.

Definition 5.1.1. *A Riemannian manifold* (X, g) *of real dimension* n *is an* asymptotically locally Euclidean (ALE) manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$ *if there exists a* *compact set $K \subset X$, and a map $\pi : X \backslash K \to \mathbb{R}^n/G$ such that for some $R > 0$ the map*

$$\pi : X \backslash K \longrightarrow (\mathbb{R}^n \backslash B_R(0))/G$$

is a diffeomorphism and such that the push-forward metric $\pi_* g$ satisfies

$$\nabla^k(\pi_* g - h_0) = O(r^{-\tau-k}) \quad \text{on} \quad \{z \in \mathbb{R}^n/G \mid r(z) > 0\}$$

for all $k \geq 0$. We denote this by saying that (X, g) is ALE$_\tau$.

Whenever we say that (X, g) is an ALE manifold asymptotic to $\mathbb{R}^n/G$ it is implicitly assumed that X is a manifold of real dimension n. We say that (X, g) is ALE if it is ALE$_\tau$ for some $\tau > 0$. The map π in Definition 5.1.1 is called an *asymptotic coordinate system* for X. Note that if a metric g is ALE$_\tau$ for some $\tau > 0$, then it is ALE$_{\tau'}$ for all $0 < \tau' < \tau$. We introduce radius functions on ALE manifolds.

Definition 5.1.2. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$. We say that a smooth function $\rho : X \to [1, \infty)$ is a radius function on X if, given an asymptotic coordinate system $\pi : X \backslash K \longrightarrow (\mathbb{R}^n \backslash B_R(0))/G$, it satisfies*

$$\nabla^k(\pi_* \rho - r) = O(r^{1-\tau-k}) \quad \text{on} \quad \{z \in \mathbb{R}^n/G \mid r(z) > R\} \tag{5.1}$$

for all $k \geq 0$.

We now present the main results of this chapter. Proper definitions of the objects and operators involved will be given in the relevant sections. We face two immediate challenges when attempting to generalize Koiso's results to ALE manifolds (X, g). The first one is that we made use of the Fredholm alternative for elliptic operators involving the linearization of the Einstein operator and the linearization of the Complex Monge-Ampère equation in the compact case. Elliptic operators on Sobolev spaces and Hölder spaces over a non-compact base manifold need not be Fredholm in general. To overcome this we use weighted versions of $C^{k,\alpha}(X)$ and $L^p_k(X)$. The second problem is that we do not have a non-compact Kodaira-Spencer type result about stability of the Kähler property for deformations of the complex structure. We will solve this by working with ALE manifolds X that arise as the complement of a smooth divisor D in a compact manifold $\bar{X}$ and use deformations of the pair $(\bar{X}, D)$ to deform the complex structure on X.

An ALE manifold (X, g) always admits a radius function ρ and it can be used to construct weighted versions of the Banach spaces $C^k(X)$, $C^{k,\alpha}(X)$ and $L^p_k(X)$. By weighting the usual norms by powers $\rho^{-\beta}$ we get Banach spaces $C^k_\beta(X)$, $C^{k,\alpha}_\beta(X)$ and $L^p_{k,\beta}(X)$, which in turn allows us to define the space $C^\infty_\beta(X)$. Weighted Sobolev and Hölder spaces are convenient for the study of ALE metrics as for instance a function $f \in C^{k,\alpha}_\beta(X)$ satisfies $f = O(\rho^\beta)$ and $\nabla^p f = O(\rho^{\beta-l})$ for all $l \leq k$. Another important aspect of the weighted spaces is that the Laplace operator is Fredholm if the weight β is sufficiently small (See Section 5.2).

Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Denote by $\mathcal{M}_{ALE_\tau}$ the space of ALE$_\tau$ metrics (Section 5.4) and denote by $\mathcal{D}_\tau$ (Definition 5.4.2) the group of diffeomorphisms on X generated by a neighbourhood of the identity map given by vector fields of small C^1-norm and appropriate control at infinity. $\mathcal{D}_{\tau+1}$ acts on $\mathcal{M}_{ALE_\tau}$ by pull-back and it introduces an equivalence relation on it. Elements of the quotient space $\mathcal{M}_{ALE_\tau}/\mathcal{D}_{\tau+1}$ are called *structures*. For each g the tangent space $T_g(\mathcal{M}_{ALE_\tau})$ can be identified with the space $C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$.

As in Section 2.3 we denote by δ^*_g the symmetric part of the covariant derivative ∇_g restricted to symmetric tensors and by δ_g the formal adjoint of δ^*_g. This makes sense for all $\tau > \frac{n}{2}$ as there is an L^2-inner product on the space $C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$ then. To study the local structure of the quotient space $\mathcal{M}_{ALE_\tau}/\mathcal{D}_{\tau+1}$ we use a slice in $\mathcal{M}_{ALE_\tau}$ for the action of $\mathcal{D}_{\tau+1}$. In the compact case the three equations $\mathrm{Ric}'_g h = 0$, $\delta_g h = 0$ and $\int_X \mathrm{tr}_g h \, dV_g = 0$ were used to define the space of infinitesimal Ricci-flat deformations $\epsilon(g)$. We will replace the two slice equations $\delta_g h = 0$ and $\int_X \mathrm{tr}_g h \, dV_g = 0$ with the *Bianchi equation* $(2\delta_g + d\,\mathrm{tr}_g)h = 0$. We do this because ALE manifolds are non-compact manifolds with infinite volume. In Section 5.4 we produce a *slice* S_τ through g in $\mathcal{M}_{ALE_\tau}$ for the action of $\mathcal{D}_{\tau+1}$ by exponentiating a neighbourhood of 0 in the space of solutions to the equation $(2\delta_g + d\,\mathrm{tr}_g)h = 0$ in $C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$.

On the set of solutions to the equation $(2\delta_g + d\,\mathrm{tr}_g)h = 0$ in $C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$ the linearization of the Einstein equation is $\nabla^*\nabla - 2\overset{\circ}{R}$. In Section 5.5 we introduce the space of *infinitesimal Ricci-flat deformations* $\epsilon^\infty_{-\tau}(g)$. It is the subspace of $C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$ defined by the two equations $(\nabla^*\nabla - 2\overset{\circ}{R})h = 0$ and $(2\delta_g + d\,\mathrm{tr}_g)h = 0$. For suitable τ the kernel of the elliptic operator $\nabla^*\nabla - 2\overset{\circ}{R}\colon C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X)) \to C^\infty_{-\tau-2}(\mathrm{Sym}^2(T^*X))$ is finite dimensional (Lemma 5.5.2), so the space $\epsilon^\infty_{-\tau}(g) \subset \ker(\nabla^*\nabla - 2\overset{\circ}{R})$ is finite dimensional.

The *moduli space* of Ricci-flat structures is the quotient space $\mathcal{R}_{ALE_\tau}/\mathcal{D}_{\tau+1}$, where $\mathcal{R}_{ALE_\tau}$ denotes the Ricci-flat metrics in $\mathcal{M}_{ALE_\tau}$. We denote by P_g the subset of S_τ of Ricci-flat metrics and call it the *premoduli space* of Ricci-flat metrics. P_g is a slice through g for the action of $\mathcal{D}_{\tau+1}$ on the space of ALE$_\tau$ Ricci-flat metrics. The real analytic implicit function theorem can be applied in this context as well and gives us the following understanding of the premoduli space of Ricci-flat metrics in the slice S_τ.

Theorem 5.1.3. *Let (X, g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$. In the slice S_τ through g there exists a finite dimensional real analytic submanifold $Z \subset S_\tau$ with $T_g Z = \epsilon^\infty_{-\tau}(g)$ and Z contains P_g as a real analytic subset.*

A complex ALE manifolds (Definition 5.3.3) is a complex manifold asymptotic to $\mathbb{C}^m/G$, where G is some finite subgroup of $U(m)$ acting freely on $\mathbb{C}^m\backslash\{0\}$. The next proposition tells us that a deformation of a Ricci-flat Kähler metric splits into a deformation of the complex structure and a deformation of a Kähler form.

Proposition 5.1.4. *Let (X, J, g) be an ALE$_\tau$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > m$. For $h \in \epsilon^\infty_{-\tau}(g)$ both its Hermitian part and its skew-Hermitian part lie in $\epsilon^\infty_{-\tau}(g)$.*

The next two theorems are the main results of this chapter. The first theorem tells us that ALE Ricci-flat deformations of ALE Ricci-flat Kähler metrics are Kähler and the second theorem tells us what the moduli space of Ricci-flat structures looks like in a neighbourhood of an ALE Ricci-flat Kähler structure. We will consider ALE Ricci-flat Kähler manifolds X that arise as the complement of a particular divisor D inside a compact Kähler manifold $\bar{X}$ as we wish to make use of the deformation theory by Kawamata for pairs $(\bar{X}, D)$ to deform the complex structure on X and prove a Kodaira-Spencer type result for deformations of the complex structure on X.

Theorem 5.1.5. *Let (X, J, g) be an ALE$_{2m}$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ and assume that $X = \bar{X}\backslash D$ for a compact Kähler manifold $(\bar{X}, \bar{J}, \bar{g})$ with a smooth and ample divisor D satisfying $K_{\bar{X}} = -\beta L_D$ for some $\beta \geq 1$. Also assume that all infinitesimal deformations of the pair $(\bar{X}, D)$ are integrable. Then small ALE$_{2m-1}$ Ricci-flat deformations of g are Kähler, possibly with respect to a perturbed complex structure.*

The motivation for considering ALE$_{2m}$ manifolds in Theorem 5.1.5 is that the low dimensional examples by Eguchi-Hanson (Examples 6.0.10) and Kronheimer ([KRO89-1] and [KRO89-2]) as well as the higher dimensional exambles by Calabi(Examples 6.0.11) all have order $2m$. Another reason for the order $2m$ in Theorem 5.1.5 is that we make use of results from [JOY00, Chapter 8] about ALE_{2m} manifolds, but also because the order $2m$ is optimal in a certain sense ([JOY00, Section 8.2.1]). A solution to $\Delta u = f$ for $f \in C^{k,\alpha}_{\beta}(X)$ for some $\beta \in (-2m, -2)$ satisfies $u \in C^{k+2,\alpha}_{\beta+2}(X)$, but if $\beta \in (-2m-1, -2m)$ then $u \in C^{k+2,\alpha}_{-2m}(X)$. The reason for the order $2m-1$ is that the cohomology of differential forms with weight $2m-1$ is related to the cohomology of the underlying manifold as explained in [JOY00, Theorem 8.4.1].

A consequence of Theorem 5.1.5 is that under the hypothesis of the theorem all infinitesimal ALE_{2m-1} Ricci-flat deformations integrate into curves of ALE_{2m-1} Ricci-flat metrics in the premoduli space P_g. P_g is almost a slice for the action of $\mathcal{D}_{2m}$ on the space of ALE_{2m-1} Ricci-flat metrics, and it can be used to study the local structure of the moduli space $\mathcal{R}_{ALE_{2m-1}}/\mathcal{D}_{2m}$.

Theorem 5.1.6. *Assume the hypothesis of Theorem 5.1.5 and assume that g is not identically flat. Then the premoduli space of ALE_{2m-1} Ricci-flat metrics is, in a neighbourhood of a Kähler structure, a smooth manifold and the moduli space of Ricci-flat structures is locally, up to an action of a finite group, a finite dimensional manifold of dimension*

$$\dim H^{1,1}_{\mathbb{R}}(X, J) - 1 + 2\dim_{\mathbb{C}} H^1(X, \mathcal{T}_X). \tag{5.2}$$

The hypothesis that g is not flat in Theorem 5.9.1 is used only in Lemma 5.9.1 to ensure that the isometry group acts as a finite group on the premoduli space. Some might wonder why there is not a term representing the anti-symmetric infinitesimal complex deformations in (5.2). This is because ALE Kähler manifolds are crepant resolutions of $\mathbb{C}^m/G$ and therefore have $H^{0,2}(X, J) = 0$. Computing the cohomology groups in Theorem 5.1.6 could be done via the compactly supported cohomology as explained in Section 5.3, but could potentially also be done by relating the cohomology groups of Theorem 5.1.6 to the cohomology groups of $\bar{X}$ and D. This is for instance done in the asymptotically cylindrical case in [KOV06]. In Example 6.0.12 we compute the cohomology groups from Theorem 5.1.6 for the pair $(\mathbb{CP}^m, \mathbb{CP}^{m-1})$.

Throughout Chapter 5 most spaces of sections of tensor bundles will have a subscript attached to them describing the order of growth of the sections. We have strived to use τ as a positive index related to an ALE_τ metric and to use the index β as a negative or general index for discussions of tensors more generally.

This chapter is organised in the following way. In Section 5.2 and Section 5.3 we provide an introduction to ALE manifolds and various known results about them. In Section 5.4 we produce a slice for the action of diffeomorphisms. In Section 5.5 we introduce ALE Ricci-flat deformations of metrics and study the corresponding space of infinitesimal deformations. In Section 5.6 we set up the theory for deformations of a pair $(\bar{X}, D)$. In Section 5.7 we discuss stability results for the deformations of the pair $(\bar{X}, D)$. In Section 5.8 and Section 5.9 we prove Theorem 5.1.5 and Theorem 5.1.6 respectively. In Section 5.10 we discuss a recent preprint on the same topic as this chapter. In Section 5.11 we discuss the possibility of extending the results advertised in this introduction from ALE manifolds to asymptotically conical manifolds.

5.2 Tools from analysis

In this section, we introduce weighted Sobolev spaces and weighted Hölder spaces and provide a number of known results about them. In Section 2.1 we introduced Sobolev spaces and Hölder spaces. Elliptic operators on Sobolev and Hölder spaces over compact manifolds have a range of useful properties. Elliptic regularity and the Fredholm alternative, just to mention a few. If we work with weighted versions of Sobolev spaces and Hölder spaces over non-compact manifolds, then, as we shall see in this section, we can recover some of these desirable properties. Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ and let ρ be a radius function on X.

Definition 5.2.1 (Weighted Sobolev spaces)**.** *Let $p, k \in \mathbb{Z}_{\geq 0}$ and let $\beta \in \mathbb{R}$. Define the weighted Sobolev space $L^p_{k,\beta}(X)$ to be the set of function $f : X \to \mathbb{R}$ which are locally integrable, k times weakly differentiable and for which the norm*

$$||f||_{L^p_{k,\beta}} = \left(\sum_{j=0}^{k} \int_X |\rho^{j-\beta} \nabla^j f|^p \rho^{-n} dV_g \right)^{1/p}$$

is finite. With this norm $L^p_{k,\beta}(X)$ is a Banach space and $L^2_{k,\beta}(X)$ is a Hilbert space.

Definition 5.2.2 (Weighted C^k spaces and Hölder spaces)**.** *Let $k \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{R}$. Define $C^k_\beta(X)$ to be the space of continuous functions $f : X \to \mathbb{R}$ with k times continuous* *derivatives and such that* $\rho^{j-\beta}|\nabla^j f|_g$ *is bounded for each* $j = 0, \ldots, k$. *Equip* $C^k_\beta(X)$ *with the norm*

$$||f||_{C^k_\beta} = \sum_{j=0}^{k} \sup_{x\in X} |\rho^{j-\beta}\nabla^j f|_g.$$

For a tensor field T *on* X *and* $\alpha, \gamma \in \mathbb{R}$ *define*

$$[T]_{\alpha,\gamma} = \sup_{x\in X, d(x,y)<\delta(g)} \left[\min(\rho(x), \rho(y))^{-\gamma} \cdot \frac{|T(x) - T(y)|_g}{d(x,y)^\alpha} \right]$$

where $\delta(g)$ *is the injectivity radius of* g *and* $d(x, y)$ *is the distance between* x *and* y *using* g. $|T(x) - T(y)|$ *is interpreted using parallel translation along a unique geodesic joining* x *and* y.

For $\beta \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$ *and* $\alpha \in (0, 1)$ *define the* weighted Hölder space $C^{k,\alpha}_\beta(X)$ *to be the set of functions* $f \in C^k_\beta(X)$ *for which the norm*

$$||f||_{C^{k,\alpha}_\beta} = ||f||_{C^k_\beta} + [\nabla^k f]_{\alpha,\beta-k-\alpha}$$

is finite. Define $C^\infty_\beta(X)$ *to be the intersection* $\cap_{k\geq 0} C^k_\beta(X)$. *Note that both* $C^k_\beta(X)$ *and* $C^{k,\alpha}_\beta(X)$ *are Banach spaces, but* $C^\infty_\beta(X)$ *is not.*

Observe that functions $f \in C^{k,\alpha}_\beta(X)$ satisfy $f = O(\rho^\beta)$ and $\nabla^j f = O(\rho^{\beta-j})$ for all $j = 0, \ldots, k$. The next Kondraschov type theorem is taken from [JOY00, Theorem 8.3.3], which in turn is an ALE version of a result by Chaljub-Simon and Choquet-Bruhat.

Theorem 5.2.3. *Let* (X, g) *be an* ALE_n *manifold asymptotic to* $\mathbb{R}^n/G$ *and let* $k \geq 0$, $\alpha \in (0, 1)$ *and* $\beta, \gamma \in \mathbb{R}$ *with* $\beta < \gamma$, *then the embedding* $C^{k,\alpha}_\beta(X) \to C^k_\gamma(X)$ *is compact.*

Similarly we borrow a Sobolev Embedding type theorem from [LEE-PAR87, Lemma 9.1]

Lemma 5.2.4 (Sobolev Embedding)**.** *Let* (X, g) *be an* ALE_n *manifold asymptotic to* $\mathbb{R}^n/G$. *Suppose* $p > 1$ *and* $l - k - \alpha > \frac{n}{p}$. *For each* $\epsilon > 0$ *there are continuous embeddings*

$$C^{l,\alpha}_{\beta-\epsilon}(X) \subset L^p_{l,\beta}(X) \subset C^{k,\alpha}_\beta(X).$$

In particular, if $f \in L^p_{l,\beta}$ *with* $l > \frac{n}{p}$ *then* $f = O(\rho^\beta)$.

Remark 5.2.5. *Let (X, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$. We wish to determine for which β and k there is an L^2-*inner product *on $C^{k,\alpha}_\beta(X)$. On $L^2_{k,\beta}(X)$ there is the inner product*

$$(\eta, \gamma)_{L^2_{k,\beta}} := \sum_{j=0}^{k} \int_X (\rho^{j-\beta}\nabla^j \eta, \rho^{j-\beta}\nabla^j\gamma)_g \rho^{-n} dV_g.$$

The radius function ρ satisfies $\rho \geq 1$, so if we assume that $\beta < -\frac{n}{2}$, then $\rho^{-2\beta-n} > 1$. Hence,

$$(\eta, \eta)_{L^2_{k,\beta}} \geq \int_X |\rho^{-\beta}\eta|^2_g \rho^{-n} dV_g = \int_X |\eta|^2_g \rho^{-2\beta-n} dV_g \geq \int_X |\eta|^2_g dV_g = (\eta, \eta)_{L^2}.$$

This shows that for $\beta < -\frac{n}{2}$ and $k > 0$ there is an L^2-inner product $(\cdot, \cdot)_{L^2}$ on $L^2_{k,\beta}(X)$. If we furthermore assume that $k > \frac{n}{2} + 1$, then Lemma 5.2.4 ensures that $C^{k,\alpha}_\beta(X) \subset L^2_{k,\beta+\epsilon}(X)$. Hence, for $k > \frac{n}{2} + 1$ and $\beta < -\frac{n}{2}$ there is an L^2-inner product on $C^{k,\alpha}_\beta(X)$. This construction extends to general tensor bundles over X.

We remark that the order of the ALE metric did not appear in Remark 5.2.5. The order of the metric has an effect on which functions belong to $C^{k,\alpha}_\beta(X)$, but it does not affect whether or not the $L^2_{k,\beta}$-inner product bounds the L^2-inner product.

The differential operators introduced in Section 2.3 act on weighted spaces in the following way. The exterior derivative d satisfies $d : C^{k,\alpha}_\beta(\wedge^r T^*X) \to C^{k-1,\alpha}_{\beta-1}(\wedge^{r+1}T^*X)$. The operator $d^*_g = (-1)^{n(r+1)+1} * d*$ acts as $d^* : C^{k,\alpha}_\beta(\wedge^r T^*X) \to C^{k-1,\alpha}_{\beta-1}(\wedge^{r-1}T^*X)$. The Levi-Civita covariant derivative ∇ satisfies $\nabla : C^{k,\alpha}_\beta(T^{(r,s)}X) \to C^{k-1,\alpha}_{\beta-1}(T^{(r-1,s)}X)$. The formal adjoint operator ∇^*_g satisfies $\nabla^* : C^{k,\alpha}_\beta(T^{(r,s)}X) \to C^{k-1,\alpha}_{\beta-1}(T^{(r-1,s)}X)$. The Riemann curvature tensor operator $(\overset{\circ}{R}\ \eta)(X_1, X_2) = \sum_{i=1}^n \eta(R(X, e_i)X_2, e_i)$ satisfies $\overset{\circ}{R}$: $C^{k,\alpha}_\beta(\mathrm{Sym}^2(T^*X)) \to C^{k-2,\alpha}_{\beta-2}(\mathrm{Sym}^2(T^*X))$.

Lemma 5.2.6 (Integration by parts)**.** *Let (X, g) be an orientable complete Riemannian manifolds with C^2 Riemann curvature tensor. Let $k > \frac{n}{2} + 2$, $\alpha \in (0, 1)$ and $\beta < -\frac{n}{2}$. Then for the L^2-inner product on $C^{k,\alpha}_\beta(\wedge^p T^*X)$ we have for any $\eta \in C^{k,\alpha}_\beta(\wedge^p T^*X)$ and $\gamma \in C^{k,\alpha}_\beta(\wedge^{p+1}T^*X)$,*

$$(d\eta, \gamma)_{L^2} = (\eta, d^*\gamma)_{L^2}$$

Proof. As $\beta < -\frac{n}{2}$ and $k > \frac{n}{2} + 2$ Remark 5.2.5 ensures the L^2-inner product is well-defined on both $C^{k,\alpha}_{\beta}(\wedge^p T^* X)$ and $C^{k-1,\alpha}_{\beta-1}(\wedge^p T^* X)$. The proof is now a direct consequence of the version of Stokes theorem for complete metrics by [GAF54]. First observe that, as in the compact case,

$$\int_X d(\eta \wedge *\gamma) = (d\eta, \gamma)_{L^2} - (\eta, d^*\gamma)_{L^2}.$$

From [GAF54, Lemma 2] we know that $|\eta \wedge *\gamma|_g \leq |\eta|_g \, |\gamma|_g$ and as $\beta < -2$ both $|\eta|_g$ and $|\gamma|_g$ are bounded. [GAF54, Theorem] then tells us that $\int_X d(\eta \wedge *\gamma) = 0$. $\square$

A similar argument to Lemma 5.2.6 can be used to show that $(\nabla \eta, \gamma)_{L^2} = (\eta, \nabla^* \gamma)_{L^2}$ for $C^{k,\alpha}_{\beta}$-sections of general tensor bundles $T^{(r,s)}X$ for $\beta < -\frac{n}{2}$ and $k > \frac{n}{2} + 2$. The next theorem is taken from [JOY00, Theorem 8.3.6] and is a generalization of results for asymptotically Euclidean manifolds by Chaljub-Simon and Choquet-Bruhat.

Theorem 5.2.7. *Let (X, g) be an ALE$_n$ manifold asympotic to $\mathbb{R}^n/G$ for $n > 2$, and ρ a radius function on X. Let $k \geq 0$ be an integer and $\alpha \in (0, 1)$. Then*

- *Let $\beta \in (-n, -2)$. Then there exists $C > 0$ such that for each $f \in C^{k,\alpha}_{\beta}(X)$ there is a unique $u \in C^{k+2,\alpha}_{\beta+2}(X)$ with $\Delta u = f$, which satisfies $||u||_{C^{k+2,\alpha}_{\beta+2}} \leq C||f||_{C^{k,\alpha}_{\beta}}$.*
- *Let $\beta \in (-n-1, -n)$. Then there exist $C_1, C_2 > 0$ such that for each $f \in C^{k,\alpha}_{\beta}(X)$ there is a unique $u \in C^{k+2,\alpha}_{-n+2}(X)$ with $\Delta u = f$. Moreover $u = A\rho^{-n+2} + v$, where*

$$A = \frac{|G|}{(n-2)\Omega_{n-1}} \int_X f \, dV_g$$

and $v \in C^{k+2,\alpha}_{\beta+2}(X)$, satisfies $|A| \leq C_1||f||_{C^0_{\beta}}$ and $||v||_{C^{k+2,\alpha}_{\beta+2}} \leq C_2||f||_{C^{k,\alpha}_{\beta}}$. Here Ω_{n-1} is the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$.

The next result about Fredholm properties of the Laplaian on ALE manifolds is related to Theorem 5.2.7 but comes from a different source. We have borrowed it from [CON-HEI13, Theorem 2.9]. They borrowed it from the PhD thesis of Stephen Marshall, who in turn borrowed it from [LOC-McO85].

Theorem 5.2.8. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Let $k \geq 0$ be an integer and let $\alpha \in (0, 1)$. Then the operator $\Delta : C^{k+2,\alpha}_{\beta+2}(X) \to C^{k,\alpha}_{\beta}(X)$ has finite dimensional kernel and closed range (i.e. is Fredholm) if $\beta + 2 \notin \{-n+2-\mathbb{N}_0\} \cup \mathbb{N}_0$.*

Weights in $\mathcal{P} = \{-n+2-\mathbb{N}_0\} \cup \mathbb{N}_0$ are called *exceptional* and weights in $\mathbb{R}\backslash\mathcal{P}$ are called *not exceptional*. We finish this section with a basic observation about ALE metrics.

Lemma 5.2.9. *Let (X, g) be an ALE$_\tau$ manifold of order $\tau > 0$. Then the metric g is complete.*

Proof. Let $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ be a Cauchy sequence on (X, g) and let X be equipped with a radius function ρ. We denote by h_0 the flat metric on $(\mathbb{R}^n\backslash B_R(0))/G$ and by r its radius function. If $\{\rho(x_i)\}_{i\in\mathbb{Z}_{\geq 0}}$ is bounded by some $R > 0$ then $\{x_i\}$ is contained in the compact set $K \cap \rho^{-1}([1, R])$, so it must converge. If $\{\rho(x_i)\}_{i\in\mathbb{Z}_{\geq 0}}$ is unbounded, then for all $\epsilon > 0$ we can find $N > 0$ and $R_0 > 0$ such that $\forall n, m \geq N$ we have $d_g(x_n, x_m) < \frac{\epsilon}{2}$ and $R_0 r(\pi_* x_n)^{-\tau} < \frac{\epsilon}{2}$. The metric g is ALE$_\tau$, so $d_{h_0}(\pi_* x_n, \pi_* x_m) = d_g(x_n, x_m) + O(r^{-\tau}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. The sequence $\{\pi_* x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is therefore Cauchy with respect to h_0. The metric h_0 is complete, so it converges to some $y \in (\mathbb{R}^n\backslash B_R(0))/G$. The sequence $\{x_i\}_{i\in\mathbb{Z}_{\geq 0}}$ therefore converges to $x = \pi^* y \in X$. □

5.3 ALE differential geometry

In this section, we introduce various differential geometric constructions on ALE manifolds. For an introduction to differential geometry on ALE manifolds we refer the reader to [JOY00, Chapter 8], where the material presented in this section is taken from.

ALE de Rham Cohomology

Let (X, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$. We denote by $H^*(X)$ the usual de Rham cohomology of X and we denote by $H^*_c(X)$ the de Rham cohomology of compactly supported forms on X. ALE manifolds can be regarded as compact manifolds with boundary S^{n-1}/G. Viewing an ALE manifold this way, the long exact sequence $\cdots \to H^k_c(X,\mathbb{R}) \to H^k(X,\mathbb{R}) \to H^k(S^{n-1}/G,\mathbb{R}) \to H^{k+1}_c(X,\mathbb{R}) \to \dots$ can be used to get the following expression for the cohomology of X.

$$\begin{aligned} &H^0(X,\mathbb{R}) = \mathbb{R} \\ &H^k(X,\mathbb{R}) = H^k_c(X,\mathbb{R}) \quad \text{for } 0 < k < n \\ &H^n(X,\mathbb{R}) = 0. \end{aligned}$$

On Weighted Hölder spaces the Hodge Laplacian $\Delta = dd^* + d^*d$ satisfies

$$\Delta : C^{k+2,\alpha}_{\beta+2}(\wedge^r T^*X) \to C^{k,\alpha}_{\beta}(\wedge^r T^*X).$$

Elements in the kernel of Δ are called harmonic forms. We denote the space of d-closed and d^*-closed r-forms by

$$(\mathcal{H}^{k,\alpha}_{\beta})^r(X) = \{\eta \in C^{k,\alpha}_{\beta}(\wedge^r T^*X) \mid d\eta = 0 \text{ and } d^*\eta = 0\}.$$

We denote by $\mathcal{H}^r_\beta(X)$ the space of smooth d-closed and d^*-closed r-forms with weight β. On a closed manifold Δ_g-harmonic forms are closed and co-closed. On non-compact manifolds this need not be true. But, as the next lemma will show, the result can be recovered over a non-compact base for weighted spaces with suitable weights β.

Proposition 5.3.1. *Let (X, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$. Let $k > \frac{n}{2} + 3$, $\alpha \in (0, 1)$ and $\beta < -\frac{n}{2}$. Then any Δ-harmonic form $\eta \in C^{k,\alpha}_{\beta}(\wedge^r T^*X)$ is d-closed and d^*-closed.*

Proof. Since $k > \frac{n}{2}+3$ and $\beta < -\frac{n}{2}$, Remark 5.2.5 says that there are L^2-inner products on $C^{k,\alpha}_{\beta}(\wedge^r T^*X)$ and $C^{k-2,\alpha}_{\beta-2}(\wedge^r T^*X)$. By Lemma 5.2.6 the operator d^* is the formal adjoint of d with respect to the L^2-inner product on $C^{k-2,\alpha}_{\beta-2}(\wedge^r T^*X)$, so the usual argument from compact manifolds applies, i.e. for η harmonic, $0 = (\Delta\eta, \eta)_{L^2} = ||d^*\eta||^2_{L^2} + ||d\eta||^2_{L^2}$. Hence $d\eta = 0$ and $d^*\eta = 0$. □

The next result is the Hodge decomposition theorem for ALE manifolds. It is taken from [JOY00, Theorem 8.4.1].

Theorem 5.3.2. *Let (X, g) be an ALE$_n$ manifold asymptotic to $\mathbb{R}^n/G$ for some $n > 2$. Then $\mathcal{H}^0_{-n+1}(X) = 0 = \mathcal{H}^n_{-n+1}(X)$ and the map $\mathcal{H}^r_{-n+1}(X) \to H^r(X, \mathbb{R}) : \eta \mapsto [\eta]$ induce isomorphisms $\mathcal{H}^r_{-n+1}(X) \simeq H^r(X, \mathbb{R}) \simeq H^r_c(X, \mathbb{R})$ for $0 < r < n$. Suppose that $-n + 1 \leq \beta < -n/2$. Then*

$$C^\infty_\beta(\wedge^r T^*X) = \mathcal{H}^r_{-n+1}(X) \otimes d\left[C^\infty_{\beta+1}(\wedge^{r-1}T^*X)\right] \oplus d^*\left[C^\infty_{\beta+1}(\wedge^{r+1}T^*X)\right]$$

where the summands are L^2-orthogonal.

ALE Kähler metrics

The model for a complex ALE manifold of complex dimension m is $(\mathbb{C}^m/G, h_0)$, where G is a finite subgroup of $U(m)$ acting freely on $\mathbb{C}^m\backslash\{0\}$ and h_0 is the standard Hermitian metric on $\mathbb{C}^m$. h_0 is invariant under the action of G, so it descends to a Hermitian metric on $\mathbb{C}^m/G$, which we also denote by h_0. Denote by r the radius function defined by h_0.

Definition 5.3.3. *Let (X, g) be a non-compact complex manifold of dimension m. We say that (X, g) is a complex ALE$_\tau$ manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > 0$ if there exists a compact subset $K \subset X$, an $R > 0$ and asymptotic coordinates(a diffeomorphism) $\pi : X\backslash K \to (\mathbb{C}^m\backslash B_R(0))/G$ such that $\nabla^k(\pi_* g - h_0) = O(r^{-\tau-k})$ and $\nabla^k(\pi_* J - J_0) = O(r^{-\tau-k})$ on $\{z \in \mathbb{C}^m/G \mid r(z) > R\}$ for all $k \geq 0$.*

Whenever we say that (X, g) is a complex ALE manifold asymptotic to $\mathbb{C}^m/G$ it is implied that X has complex dimension m. For the purpose of this project a *resolution* (X, π) of a singularity $\mathbb{C}^m/G$ is a nonsingular manifold X and a proper birational map $\pi : X \to \mathbb{C}^m/G$. It follows from [JOY00, Section 8.9] that an ALE$_{2m}$ Kähler manifold (X, π) is birational to a deformation of $\mathbb{C}^m/G$. However, for $m \geq 3$ the Schlessinger Rigidity Theorem ([JOY00, Theorem 6.4.8]) asserts that $\mathbb{C}^m/G$ is rigid, so (X, π) is in fact a resolution of $\mathbb{C}^m/G$. It follows from [JOY00, Proposition 8.2.1] and [JOY00, Theorem 8.2.4] that a resolution (X, π) of $\mathbb{C}^m/G$ for a finite subgroup $G \subset U(m)$ acting freely on $\mathbb{C}^m\backslash\{0\}$ with an ALE$_{2m}$ Ricci-flat Kähler metric g is in fact a *crepant resolution*, i.e. $K_X \simeq \pi^* K_{\mathbb{C}^m/G}$, and $G = SU(m)$. By [JOY00, Theorem 8.4.3] such a crepant resolution satisfies $H^{2,0}(X) = H^{0,2}(X) = 0$. By [JOY00, Proposition 8.4.5] ALE$_{2m}$ Kähler manifolds admit in each Kähler class an ALE$_{2m}$ Kähler metric which is flat, i.e. $g = \pi^* h_0$, outside a compact set.

There are weighted Banach spaces of (p, q)-forms $L^l_{k,\beta}(\wedge^{p,q}X)$ and $C^{k,\alpha}_\beta(\wedge^{p,q}X)$ on ALE Kähler manifolds and we have the following splitting theorem borrow from [JOY00, Theorem 8.4.2]

Theorem 5.3.4. *Let (X, J, g) be an ALE$_{2m}$ Kähler manifold asymptotic to $\mathbb{C}^m/G$. Define $\mathcal{H}^{p,q}(X) = \{\eta \in C^\infty_{-2m+1}(\wedge^{p,q}X) \mid d\eta = d^*\eta = 0\}$. Then $\mathcal{H}^{p,q}(X)$ is a finite dimensional vector space, and the map from $\mathcal{H}^{p,q}(X)$ to $H^{p+q}(X, \mathbb{C})$ defined by $\eta \mapsto [\eta]$ is injective. Define $H^{p,q}(X)$ to be the image of this map. Then $H^k(X, \mathbb{C}) = \oplus_{j=0}^k H^{j,k-j}(X)$ for $0 < k < 2m$.*

Lemma 5.3.5. *Let (X, J, g) be a complex ALE Kähler manifold asymptotic to $\mathbb{C}^m/G$. Let $k > 2m + 3$ be an integer and let $\beta < -m$. Then any $\Delta_{\bar\partial}$-harmonic form in $C^{k,\alpha}_\beta(\wedge^{0,r}X)$ is $\bar\partial$- and $\bar\partial^*$-closed.*

Proof. Let η be a real form in $\ker(\Delta_{\bar\partial}) \cap C^{k,\alpha}_\beta(\wedge^{0,r}T^*X)$. On Kähler manifolds $\Delta_{\bar\partial}\eta = \frac{1}{2}\Delta_g\eta = 0$, and so for $\eta \in \ker(\Delta_{\bar\partial}) \cap C^{k,\alpha}_\beta(\wedge^{0,r}(T^*X)$, Lemma 5.3.1 implies that $d\eta = 0$ and $d^*\eta = 0$. On a complex manifold $d = \partial + \bar\partial$ and $d^* = \partial^* + \bar\partial^*$, so $\bar\partial\eta = 0$ and $\bar\partial^*\eta = 0$. □

We borrow a global $\partial\bar\partial$-lemma from [JOY00, Theorem 8.4.4]

Theorem 5.3.6. *Let $m > 1$ and let (X, J, g) be an ALE$_{2m}$ Kähler manifold asymptotic to $\mathbb{C}^m/G$. Let $\beta < -m$ and suppose that $\eta \in C^\infty_\beta(\wedge^{1,1}_{\mathbb{R}}X)$ is a closed real $(1,1)$-form and $[\eta] = 0$ in $H^2(X, \mathbb{R})$. Then there exists a unique real function $u \in C^\infty_{\beta+2}(X)$ with $\eta = \partial\bar\partial u$.*

5.4 Slice construction

In this section, we construct a slice for the action of a group of diffeomorphisms on the space of ALE metrics on an ALE manifold. A *slice* through a point is a subset containing the point such that for each orbit passing through the subset exactly one element of the orbit lies in the subset. It is used as a chart on the orbit space. Koiso used in [KOI83] the two equations $\delta_g h = 0$ and $\int \mathrm{tr}_g h = 0$ to construct a slice for the action of the group of diffeomorphisms on the volume 1 metrics on a compact manifold. ALE manifolds have infinite volume, so it will be convenient to replace to two equations $\delta_g h = 0$ and $\int \mathrm{tr}_g h = 0$ with the single equation $(2\delta_g + d\,\mathrm{tr}_g)h = 0$. This slice equation has been used before by various people, e.g. Biquard([BIQ00]), Kovalev([KOV06]) and Nordström([NOR08]).

The space $\mathcal{M}_{ALE_\tau}$

Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ or order $\tau > 0$. Denote by $\mathcal{M}$ the collection of complete Riemannian metrics in $C^\infty(\mathrm{Sym}^2(T^*X))$. We give it the subspace topology. ALE$_\tau$ metrics belong to $\mathcal{M}$ by Lemma 5.2.9. Denote by $\mathcal{M}_{ALE}$ and $\mathcal{M}_{ALE_\tau}$ the subspaces of $\mathcal{M}$ of ALE and ALE$_\tau$ metrics respectively and equip them both with a subspace topology.

Let ρ be a radius function on the ALE$_\tau$ manifold X and let π be the asymptotic coordinate system. Take $\beta \in \mathbb{R}$. A function $f \in C^\infty(X)$ satisfies $f = O(\rho^\beta)$ exactly if $\pi_* f = O((\pi_*\rho)^\beta)$, which happen exactly if $\pi_* f = O(r^\beta)$. Also $|\nabla^k_{\pi_* g}\pi_* f - \nabla^k_{h_0}\pi_* f|_{h_0} = O(r^{-\tau-k})$. Hence $f \in C^\infty_{-\tau}(X)$ exactly if $|\nabla^k_{h_0}\pi_* f|_{h_0} = O(r^{-\tau-k})$. The same holds for tensors. If $h \in C^\infty(\mathrm{Sym}^2(T^*X))$ with $|h|_g < 1$ satisfies $|\nabla^k_{h_0} h|_{h_0} = O(r^{-\tau-k})$ then $g+h$ is ALE$_\tau$. The tangent space to $\mathcal{M}_{ALE_\tau}$ at g is therefore described as $T_g(\mathcal{M}_{ALE_\tau}) = C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$. Similarly, for any two ALE$_\tau$ metrics g', g'' the difference $h = g' - g''$ is an element of $C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$. We say that a deformation g_t of g is *small* if $|h_t|_g$, where $h_t = g_t - g$, is sufficiently small.

Lemma 5.4.1. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Then for any $\tau' \leq \tau$, small ALE$_{\tau'}$ deformations of g are ALE$_{\tau'}$.*

This follows from observing that for an ALE$_\tau$ metric g a deformation $g+h$ of g with $h \in C^\infty_{-\tau'}(\mathrm{Sym}^2(T^*X))$ satisfies $\nabla^k(\pi_*(g+h) - h_0) = \nabla^k(\pi_* g - h_0) + \nabla^k(\pi_* h) = O(r^{-\tau-k}) + O(r^{-\tau'-k}) = O(r^{-\tau'-k})$.

We define the space $\mathcal{M}_{k,\tau} = L^2_{k,-\tau}(\mathrm{Sym}^2(T^*X)) \cap \mathcal{M}_{ALE_\tau}$. This definition makes sense as any $g' \in \mathcal{M}_{ALE_\tau}$ can be written as $g+h$ for some $h \in L^2_{k,-\tau}(\mathrm{Sym}^2(T^*X))$ and g' satisfies $\nabla_g g' = \nabla_g(h-g) = \nabla_g h$. Hence $||g'||_{L^2_{k,-\tau}} = ||h||_{L^2_{k,-\tau}}$. The tangent space $T_g\mathcal{M}_{k,\tau}$ is naturally identified with $L^2_{k,-\tau}(\mathrm{Sym}^2(T^*X))$. Also define $\mathcal{M}^{k,\alpha}_{ALE_\tau}$ and $\mathcal{M}^k_{ALE_\tau}$ to be the ALE$_\tau$ metrics in $C^{k,\alpha}_{-\tau}(\mathrm{Sym}^2(T^*X))$ and $C^k_{-\tau}(\mathrm{Sym}^2(T^*X))$ respectively. Same argument as for $\mathcal{M}_{k,\tau}$ shows that these definitions make sense. The tangent space $T_g\mathcal{M}^{k,\alpha}_{ALE_\tau}$ is naturally identified with $C^{k,\alpha}_{-\tau}(\mathrm{Sym}^2(T^*X))$ and the tangent space $T_g\mathcal{M}^k_{ALE_\tau}$ is naturally identified with $C^k_{-\tau}(\mathrm{Sym}^2(T^*X))$.

Diffeomorphisms

Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$. Denote by δ the *injectivity radius* of X. Note that for ALE manifolds the injectivity radius is always strictly positive. Denote by $\mathcal{D} = \mathcal{D}(X)$ the group of diffeomorphisms from X to itself and equip it with the supremum norm $||\phi||_\infty = \sup_{x\in X} \mathrm{dist}_g(x, \phi(x))$. We say that $\phi \in \mathcal{D}$ is *given by* $V \in C^\infty(TX)$, with $||V||_{C^1}$ sufficiently small, if $\phi(p) = exp_p(V_p)$ for each $p \in X$.

Definition 5.4.2. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ or order $\tau > 0$. Let $\delta > 0$ be the injectivity radius of X. Denote by $\mathcal{D}_\tau$ the group of diffeomorphisms* *generated by the neighbourhood of the identity map defined by those diffeomorphisms ϕ which are given by some $V \in C^{\infty}_{-\tau}(TX)$ with $|V|_g \leq \delta$.*

A similar group of diffeomorphisms has been used by Kovalev in [KOV06]. The weight of the generating vector field V of a diffeomorphism $\phi \in \mathcal{D}_{\tau+1}$ in Definition 5.4.2 ensures that ϕ satisfies $|\nabla^k_{h_0}(\pi_*(\phi^* - Id^*)g')|_{h_0} = O(r^{-\tau-k})$ for all $g' \in \mathcal{M}_{ALE_\tau}$ and all $k \geq 0$. This gives us the next lemma.

Lemma 5.4.3. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. For any $\phi \in \mathcal{D}_{\tau+1}$ and any $g' \in \mathcal{M}_{ALE_\tau}$ the metric $\phi^* g'$ is $\mathcal{M}_{ALE_\tau}$.*

Proof. For all $k \geq 0$ the metric $\phi^* g'$ satisfies

$$\begin{aligned}
|\nabla^k_{h_0}(\pi_*\phi^* g' - h_0)|_{h_0} &= |\nabla^k(\pi_*\phi^* g') - \nabla^k h_0| \\
&\leq |\nabla^k(\pi_*\phi^* g') - \nabla^k(\pi_* g')| + |\nabla^k(\pi_* g') - \nabla^k h_0| \\
&= |\nabla^k(\pi_*(\phi^* - Id^*)g')| + |\nabla^k(\pi_* g' - h_0)| \\
&= O(r^{-\tau-k}) + O(r^{-\tau-k}) \\
&= O(r^{-\tau-k}).
\end{aligned}$$

The metric $\phi^* g'$ is therefore ALE$_\tau$. ☐

We explained in Section 2.3 that a 1-parameter flow ϕ_t of diffeomorphisms with infinitesimal generator $X \in C^{\infty}(TX)$ satisfies $\mathcal{L}_X g = \frac{d}{dt}(\phi_t^* g)|_{t=0}$. If the flow ϕ_t belongs to $\mathcal{D}_{\tau+1}$, then $T_g((\mathcal{D}_{\tau+1})^* g) \subset T_g(\mathcal{M}_{ALE_\tau})$.

Define $\delta^* : C^{\infty}_{\beta}(\mathrm{Sym}^k(T^*X)) \to C^{\infty}_{\beta-1}(\mathrm{Sym}^{k+1}(T^*X)) : \eta \mapsto \mathrm{Sym} \circ \nabla|_{\mathrm{Sym}^k(T^*X))}$, where $\beta \in \mathbb{R}$. For $\beta < -\frac{n}{2}$, δ^* admits an L^2-formal adjoint δ given by $\delta = \nabla^*|_{\mathrm{Sym}^{k+1}(T^*X)}$. This is similar to the construction in Section 2.3. From [BES87, Lemma 1.60] we get the expression $\delta^*\eta = \frac{1}{2}\mathcal{L}_{\eta^\#} g$ for a 1-form $\eta \in C^{\infty}_{\beta}(T^*X)$, where $\beta < -\frac{n}{2}$ and $\eta^\#$ is the dual vector field. The tangent space of the orbit $(\mathcal{D}_{\tau+1})^* g$ at g can therefore be described as $T_g((\mathcal{D}_{\tau+1})^* g) = \mathrm{Im}(\delta^*) \cap T_g(\mathcal{M}_{ALE_\tau})$.

Slice equation

Proposition 5.4.4. *Let (X, g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Let $k > \frac{n}{2} + 3$, $\alpha \in (0, 1)$ and $\beta < -\frac{n}{2}$. Then for $\eta \in C^{k,\alpha}_{\beta}(\wedge^1 T^*X)$, $\Delta\eta = \nabla^*\nabla\eta$. Also $\Delta\eta = 0$ exactly if $\nabla\eta = 0$.*

Proof. The first statement follows immediately from the Weitzenböck formula for 1-forms, (2.4). The second statement follows from integration by parts, Lemma 5.2.6, as $(\Delta\eta, \eta)_{L^2} = (\nabla^*\nabla\eta, \eta)_{L^2} = (\nabla\eta, \nabla\eta)_{L^2} = ||\nabla\eta||^2$. □

Proposition 5.4.5. *Let (X, g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Let $k > \frac{n}{2} + 3$, $\alpha \in (0, 1)$ and $\beta < -\frac{n}{2}$. Then on the space $C^{k,\alpha}_\beta(\wedge^1 T^*X)$ it holds that $\Delta = (2\delta_g + d\, tr_g)\delta^*$.*

Proof. The symmetric part of $\nabla : C^{k,\alpha}_\beta(\wedge^1 T^*X) \to C^{k-1,\alpha}_{\beta-1}(\wedge^2 T^*X)$ is δ^* and on 1-forms the anti-symmetric part is $\frac{1}{2}d$. So for $\alpha \in C^{k,\alpha}_\beta(\wedge^1 T^*X)$ we have $\delta^*\alpha = \nabla\alpha - \frac{1}{2}d\alpha$ for a 1-form α. Also, $\mathrm{tr}_g \circ \delta^*_g = d^*$. Using Proposition 5.4.4 we get

$$(2\delta + d\,\mathrm{tr}_g)\delta^* = 2\nabla^*\nabla - \nabla^* d + d\,\mathrm{tr}_g\delta^* = 2\nabla^*\nabla - d^*d - dd^* = \Delta_g.$$

□

Proposition 5.4.6. *Let (X, g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Let $k > \frac{n}{2} + 3$, $\alpha \in (0, 1)$ and $\beta < -\frac{n}{2}$. Then*

$$C^{k,\alpha}_\beta(Sym^2(T^*X)) = Im(\delta^*_g) \oplus \ker(2\delta_g + d\, tr_g),$$

where $\delta^*_g : C^{k+1,\alpha}_{\beta+1}(\wedge^1 T^*X) \to C^{k,\alpha}_\beta(Sym^2(T^*X))$ and $2\delta_g + d\, tr_g : C^{k,\alpha}_\beta(Sym^2(T^*X)) \to C^{k-1,\alpha}_{\beta-1}(\wedge^1 T^*X)$.

Proof. Ricci-flatness of g and the weight β allows us to apply Proposition 5.4.4, so harmonic 1-forms in $C^{k+1,\alpha}_{\beta+1}(\wedge^1 T^*X)$ are parallel and closed. On such 1-forms $\delta^* = \nabla - \frac{1}{2}d$. For a harmonic form $\alpha \in C^{k+1,\alpha}_{\beta+1}(\wedge^1 T^*X)$ we therefore have

$$\delta^*\alpha = (\nabla - \frac{1}{2}d)\alpha = \nabla\alpha - \frac{1}{2}d\alpha = 0 - 0 = 0,$$

which by contraposition implies that if an $\alpha \in C^{k+1,\alpha}_{\beta+1}(\wedge^1 T^*X)$ satisfies $\delta^*\alpha \neq 0$ then $\Delta_g\alpha \neq 0$. Using the relation $\Delta = (2\delta + d\,\mathrm{tr}_g)\delta^*$ on 1-forms from Proposition 5.4.5 we conclude that non-zero elements in $\mathrm{Im}(\delta^*)$ are not in the kernel of $2\delta + d\,\mathrm{tr}_g$.

Take $h \in C^{k,\alpha}_\beta(\mathrm{Sym}^2(T^*X))$. Then $(2\delta_g + d\,\mathrm{tr}_g)h \in C^{k-1,\alpha}_{\beta-1}(X)$. Theorem 5.2.7 then provides a solution u to the equation $\Delta u = (2\delta_g + d\,\mathrm{tr}_g)h$. Now, $(2\delta_g + d\,\mathrm{tr}_g)(\delta^*_g u - h) = \Delta u - (d\delta_g + d\,\mathrm{tr}_g)h = \Delta u - \Delta u = 0$, so the element $\delta^*_g u - h$ satisfies $\delta^*_g u - h \in$

$\ker(2\delta_g + d\,\mathrm{tr}_g)$. Hence we have arrived at the desired decomposed $h = \delta_g^* u + (\delta_g^* u - h)$, where h has been decomposed into a sum of an element from $\mathrm{Im}(\delta_g^*)$ and and element from $\ker(2\delta_g + d\,\mathrm{tr}_g)$. □

For an ALE$_\tau$ manifold (X, g) asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$ we call the equation $(2\delta_g + d\,\mathrm{tr}_g)h = 0$ for the *slice equation* for the action of the diffeomorphism group $\mathcal{D}_{\tau+1}$ as solutions to it in $C^{k,\alpha}_{-\tau}(\mathrm{Sym}^2(T^*X))$ are orthogonal to the tangent space of the orbit $(\mathcal{D}_{\tau+1})^* g$ at g. In the next subsection we will see that exponentiating a neighbourhood of 0 in $\ker(2\delta + d\,\mathrm{tr}_g) \subset C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$ produces a slice through g in $\mathcal{M}_{ALE_\tau}$ for the action of $\mathcal{D}_{\tau+1}$.

Slice construction

Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. In this subsection, we produce a slice for the action of $\mathcal{D}_{\tau+1}$ on the space $\mathcal{M}_{ALE_\tau}$. Denote by $\mathcal{D}^{k,\alpha}_\tau$ the subgroup of $\mathcal{D}_\tau$ consisting of those diffeomorphisms which are $C^{k,\alpha}$. Denote by $\mathcal{M}^{k,\alpha}_{ALE_\tau}$ the space $\mathcal{M}_{ALE_\tau} \cap C^{k,\alpha}(\mathrm{Sym}^2(T^*X))$. We have the following action

$$\mathcal{D}^{k+1,\alpha}_{\tau+1} \times \mathcal{M}^{k,\alpha}_{ALE_\tau} \to \mathcal{M}^{k,\alpha}_{ALE_\tau} : (\phi, g') \mapsto \phi^* g'.$$

Denote by $I^{k,\alpha}_\tau(g)$ the isometry group $\{\eta \in \mathcal{D}^{k,\alpha}_\tau \mid \eta^* g = g\}$.

Theorem 5.4.7 (Slice theorem, $C^{k,\alpha}_\tau$ version)**.** *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$ and let $k > \frac{n}{2} + 3$. Then there exists a subset $S^{k,\alpha}_\tau \subset \mathcal{M}^{k,\alpha}_{ALE_\tau}$ through g satisfying*

i) If $\eta \in I^{k+1,\alpha}_{\tau+1}(g)$ then $\eta^*(S^{k,\alpha}_\tau) = S^{k,\alpha}_\tau$.

ii) If $\eta \in \mathcal{D}^{k+1,\alpha}_{\tau+1}$ satisfies $\eta^* S^{k,\alpha}_\tau \cap S^{k,\alpha}_\tau \neq \emptyset$ then $\eta \in I^{k+1,\alpha}_{\tau+1}(g)$.

iii) There exists a local cross section $\chi : \mathcal{D}^{k+1,\alpha}_{\tau+1}/I^{k+1,\alpha}_{\tau+1}(g) \to \mathcal{D}^{k+1,\alpha}_{\tau+1}$ defined in a neighbourhood U of the identity coset such that

$$F : U \times S^{k,\alpha}_\tau \to \mathcal{M}^{k,\alpha}_{ALE_\tau} : (u, t) \mapsto \chi(u)^* t$$

is a homeomorphism onto a neighbourhood of g.

Proof. The construction of the local cross section χ in [EBI70, Proposition 5.10] generalizes to ALE$_\tau$ manifolds to produce a local cross section $\chi : \mathcal{D}^{k+1,\alpha}_{\tau+1}/I^{k+1,\alpha}_{\tau+1}(g) \to \mathcal{D}^{k+1,\alpha}_{\tau+1}$. Denote by $\mathcal{O}^{k+1,\alpha}_{\tau+1}(g)$ the orbit $(\mathcal{D}^{k+1,\alpha}_{\tau+1})^*g = (\mathcal{D}^{k+1,\alpha}_{\tau+1}/I^{k+1,\alpha}_{\tau+1}(g))^*g$. It is a submanifold of the Banach manifold $\mathcal{M}^{k,\alpha}_{ALE_\tau}$. Denote by ν the smooth normal bundle to the tangent bundle of $\mathcal{O}^{k+1,\alpha}_{\tau+1}(g)$. By Proposition 5.4.6 fibres are given by $\nu_{g'} = \ker(2\delta^*_{g'} + 2d\mathrm{tr}_{g'}) \cap C^{k,\alpha}_{-\tau}(\mathrm{Sym}^2(T^*X))$.

Let $U' \subset \mathcal{D}^{k+1,\alpha}_{\tau+1}/I^{k+1,\alpha}_{\tau+1}(g)$ be a neighbourhood of the identity coset small enough that it admits a cross section $\chi : U' \to \mathcal{D}^{k+1,\alpha}_{\tau+1}$ and let $\epsilon' > 0$ and U' be chosen such that the following is satisfied. Let $V' = \{v \in \nu_g \,|\, ||v||_g \leq (\epsilon')^2\}$ and let $W' = \{d(\eta^*)v \,|\, v \in V' \text{ and } \eta \in \chi(U')\}$ be such that $\exp|_{W'}$ is a diffeomorphism onto its image and $\exp(W') \cap O^{k+1,\alpha}_{\tau+1}(g) = U'$. For some $\delta > 0$ we have $B_{2\delta}(g) \subset \exp(W')$. Now choose $U \subset U'$ and $\epsilon < \epsilon'$ such that $\exp(W) \subset B_\delta(g)$. Define $S^{k,\alpha}_\tau = \exp(V)$.

The set $S^{k,\alpha}_\tau$ has the three properties of a slice. The argument is similar to the orbifold case. See the proof of Claim 3.6.5 for details. □

Theorem 5.4.8 (Slice theorem, C^∞_τ version)**.** *If we replace $\mathcal{D}^{k+1,\alpha}_{\tau+1}$ by $\mathcal{D}_{\tau+1}$ and $\mathcal{M}^{k,\alpha}_{ALE_\tau}$ by $\mathcal{M}_{ALE_\tau}$ in the hypothesis of Theorem 5.4.7, then there exists a slice $S_\tau \subset \mathcal{M}_{ALE_\tau}$ with the three properties of a slice outlines in Theorem 5.4.7.*

Proof. Generalizing the proof of Theorem 5.4.7 to the smooth case is similar to the proof of [EBI70, Theorem 7.4]. The details of the proof of Theorem 3.6.6 applies to the case of ALE$_\tau$ manifolds. □

From Theorem 5.4.8 it follows that

Corollary 5.4.9. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$. Then the slice $S_\tau \subset \mathcal{M}_{ALE_\tau}$ through g constructed in Theorem 5.4.8 produces a homeomorphism*

$$S_\tau/I_{\tau+1}(g) \to U \subset \mathcal{M}_{ALE_\tau}/\mathcal{D}_{\tau+1}$$

where U is a neighbourhood of the identity coset $[g]$.

Denote by $P_g \subset S_\tau$ the subspace of Ricci-flat metrics in S_τ and call it the *premoduli space* of Ricci-flat metrics. Denote by $\mathcal{R}_{ALE_\tau}$ the subspace of $\mathcal{M}_{ALE_\tau}$ of Ricci-flat metrics.

Corollary 5.4.10. *Let (X, g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$, then the homeomorphism of Proposition 5.4.9 restricts to a homeomorphism*

$$P_g/I_{\tau+1}(g) \to U \subset \mathcal{R}_{ALE_\tau}/\mathcal{D}_{\tau+1}.$$

5.5 ALE Ricci-flat deformations

In this section we study ALE$_\tau$ Ricci-flat deformations on a non-compact manifold X. We introduce the space of infinitesimal Ricci-flat deformations and study the Hermitian and skew-Hermitian components of it.

Let (X, g) be a Riemannian manifold. A *family of deformations* of g is a smooth curve $(-\epsilon, \epsilon) \to C^\infty(\mathrm{Sym}^2(T^*X)) : t \mapsto g_t$ with $g_0 = g$. The deformation g_t can be written as $g_t = g + h_t$ for $h_t \in C^\infty(\mathrm{Sym}^2(T^*X))$. We say that a family of deformations is ALE$_\tau$, Einstein, Ricci-flat, Kähler etc. if each deformation is ALE$_\tau$, Einstein, Ricci-flat, Kähler respectively.

Infinitesimal Ricci-flat deformations

Let (X, g) be an ALE_τ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$. In the last section we constructed for $k > \frac{n}{2} + 3$ a slice $S^{k,\alpha}_\tau \subset \mathcal{M}_{ALE_\tau}$ for the pull-back action by diffeomorphisms. Consider an ALE_τ Ricci-flat deformation g_t of smooth Ricci-flat metrics of g. Then $h = \frac{d}{dt} g_t|_{t=0} \in C^\infty_{-\tau}(\mathrm{Sym}^2(T^*X))$ satisfies $\mathrm{Ric}'_g(h) = 0$. Assuming that the curve g_t belongs to $S^{k,\alpha}_\tau$, then h belongs to $T_g S^{k,\alpha}_\tau$ and satisfies the equation $(2\delta_g + d\,\mathrm{tr}_g)h = 0$. This inspires the next definition.

Definition 5.5.1 (Infinitesimal Ricci-flat deformations)**.** *Let (X, g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. For an integer $k > 2$ and $\alpha \in (0, 1)$ define the space $\epsilon^{k,\alpha}_{-\tau}(g) \subset C^{k,\alpha}_{-\tau}(Sym^2(T^*X))$ by the equations*

$$Ric'_g(h) = 0, \qquad (2\delta_g + d\,tr_g)h = 0. \tag{5.3}$$

Similarly define $\epsilon^\infty_{-\tau}(g)$ to the be solutions to (5.3) in $C^\infty_{-\tau}(Sym^2(T^*X))$. We call $\epsilon^\infty_{-\tau}(g)$ the space of infinitesimal Ricci-flat deformations of g.

Lemma 5.5.2. *Let (X,g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$. Let $k > \frac{n}{2} + 3$ and $\alpha \in (0,1)$. Then* $\dim \epsilon^{k,\alpha}_{-\tau}(g) < \infty$.

Proof. Towards infinity the curvature R approximates zero, so the elliptic operator $\Delta_L = \nabla^*\nabla - 2 \overset{\circ}{R}$ approximates $\nabla^*\nabla$, and Δ via the Weitzenböck formula. Choose an $\epsilon \in (0,1)$ such that $-\tau+\epsilon$ is not exceptional and denote by $L^2_{k,-\tau+\epsilon}$ the space $L^2_{k,-\tau+\epsilon}(\text{Sym}^2(T^*X))$. It then follows from [CAN81, Theorem 7.4] that $\Delta_L|_{L^2_{k,-\tau+\epsilon}}$ has a finite dimensional kernel. From the inclusion $\epsilon^{k,\alpha}_{-\tau}(g) \subset C^{k,\alpha}_{-\tau}(\text{Sym}^2(T^*X)) \subset L^2_{k,-\tau+\epsilon}(\text{Sym}^2(T^*X))$ it then follows that $\dim \epsilon^{k,\alpha}_{-\tau}(g) < \infty$. □

We remark that the question about the finite dimensionality of $\epsilon^{k,\alpha}_{-\tau}(g)$ in Lemma 5.5.2 is independent of k, α and τ, provided k and τ are sufficiently large. The space $\epsilon^{\infty}_{-\tau}(g)$ is contained in $\epsilon^{k,\alpha}_{-\tau}(g)$ so it follows from Lemma 5.5.2 that $\epsilon^{\infty}_{-\tau}(g)$ is finite dimensional. For the convenience of the reader we restate Theorem 5.1.3.

Theorem 5.5.3. *Let (X,g) be an ALE$_\tau$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$, for $\tau > \frac{n}{2}$. Let S_τ be the slice through g in $\mathcal{M}_{ALE_\tau}$ for the action of the group of diffeomorphisms $\mathcal{D}_{\tau+1}$. Then there exists a finite dimensional real analytic submanifold $Z \subset S_\tau$ with $T_gZ = \epsilon^{\infty}_{-\tau}(g)$ and Z containing P_g as a real analytic subset.*

Proof. Let $k \geq \frac{n}{2} + 3$ and let $S^{k,\alpha}_\tau$ be a slice through g in $\mathcal{M}^{k,\alpha}_{ALE_\tau}$. Restrict the Ricci operator Ric to this slice, i.e Ric : $S^{k,\alpha}_\tau \subset \mathcal{M}^{k,\alpha}_{ALE_\tau} \to C^{k-2,\alpha}_{-\tau-2}(\text{Sym}^2(T^*X))$. Choose an $\epsilon \in (0,1)$ such that $-\tau+\epsilon$ is not exceptional. The construction of a slice in Theorem 5.4.7 could equally well have been proved for Sobolev spaces instead of Hölder spaces. We can therefore consider the slice $S^2_{k,-\tau+\epsilon}$ of $L^2_{k,-\tau+\epsilon}$-metrics though g. The elliptic operator $F = 2\text{Ric}'_g + 2\delta^*\delta = \nabla^*\nabla - 2 \overset{\circ}{R} - \nabla d\,\text{tr}_g$ on the Hilbert space $L^2_{k,-\tau+\epsilon}(\text{Sym}^2(T^*X))$ has closed range by [CAN81, Theorem 7.4] and it follows from a similar argument to [BES87, Lemma 12.48] that $F(T_gS^2_{k,-\tau+\epsilon})$ is closed as well. The operator Ric satisfies $\text{Ric}'_g(T_gS^2_{k,-\tau+\epsilon}) = F(T_gS^2_{k,-\tau+\epsilon})$ and it is real analytic by [DeT-KAZ81]. We can therefore apply Theorem 2.1.11 to conclude that there exists a neighbourhood U of g in $S^2_{k,-\tau+\epsilon}$ such that $U \cap (P_g)^2_{k,-\tau+\epsilon}$ is a real analytic subset of a real analytic submanifold $Z^2_{k,-\tau+\epsilon} \subset U$ whose tangent space $T_gZ^2_{k,-\tau+\epsilon}$ coincides with $\ker(\text{Ric}'_g) \cap T_gS^2_{k,-\tau+\epsilon} = \epsilon^2_{k,-\tau+\epsilon}(g)$. Intersecting this by $C^{k,\alpha}_{-\tau}(\text{Sym}^2(T^*X))$ gives us a real analytic submanifold $Z^{k,\alpha}_{-\tau}$ of $S^{k,\alpha}_\tau$ with $T_gZ^{k,\alpha}_{-\tau} = \epsilon^{k,\alpha}_{-\tau}(g)$ which contains $U \cap (P_g)^{k,\alpha}_{-\tau}$ as a real analytic subset. By Lemma 5.5.2 the space $\epsilon^{k,\alpha}_{-\tau}(g)$ is finite dimensional. This holds for arbitrarily large k. In particular, it holds for the slice S_τ. □

Let (X, g) be a Ricci-flat ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > \frac{n}{2}$. From Theorem 5.5.3 we know that every $h \in \epsilon^{\infty}_{-\tau}(g)$ integrates into a curve of ALE$_\tau$ metrics through g in the slice S_τ. Similarly to the compact case, we wish to identify conditions on X and g for which we can be sure that any $h \in \epsilon^{\infty}_{-\tau}(g)$ integrates into a curve of ALE$_\tau$ Ricci-flat metrics through g. Note that if each $h \in \epsilon^{\infty}_{-\tau}(g)$ is the infinitesimal deformation of a curve in $\mathcal{M}_{ALE_\tau}$ through g of Ricci-flat metrics, then the premoduli space P_g of ALE$_\tau$ Ricci-flat metric spans an entire neighbourhood of g in S_τ.

In proving Theorem 5.1.5 we will be working with Ricci-flat Kähler metrics which are of class at least C^2, so they are actually smooth by [DeT-KAZ81, Theorem 6.1]. We will therefore not limit ourselves by restricting our attention to $\epsilon^{\infty}_{-\tau}(g)$ rather than bigger spaces like $\epsilon^{k,\alpha}_{-\tau}(g)$.

From now on let (X, J, g) be an ALE$_\tau$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ for a finite subgroup $G \subset U(m)$ of order $\tau > m$. Any symmetric $(2, 0)$-tensor $h \in C^{\infty}(\mathrm{Sym}^2(T^*X))$ splits as $h = h_H + h_A$, where h_H is *Hermitian* symmetric and h_A is *skew-Hermitian* symmetric, that is $h_H(JX, JY) = h_H(X, Y)$ and $h_A(JX, JY) = -h_A(X, Y)$. The defining equations of $\epsilon^{\infty}_{-\tau}(g)$ for Hermitian and skew-Hermitian tensors respectively can be reformulated in a convenient way. For appropriately chosen weights this is similar to the compact case (and the orbifold case), so we keep it short to minimize repetition. Note that if $h = O(r^{-\tau})$, then $h \circ J = O(r^{-\tau})$ as

$$\begin{aligned} h \circ J(X) &= (h \circ J \circ \pi^{1,0} + h \circ J \circ \pi^{0,1})(X^{1,0} + X^{0,1}) \\ &= h \circ J|_{T^{1,0}X}(X^{1,0}) + h \circ J|_{T^{0,1}X}(X^{0,1}) = ih(X^{1,0}) + (-i)h(X^{0,1}) \\ &= O(r^{-\tau}) + O(r^{-\tau}) = O(r^{-\tau}). \end{aligned}$$

The complex structure J cannot belong to $C^{\infty}_{-\tau}(TX \otimes T^*X)$ for any $\tau > 0$, as J satisfies $(J^2)^i{}_j = J^i{}_k J^k{}_j = \delta_{ij}$. If $|J^i{}_j| \to 0$ at any rate, then so does $|J^i{}_k J^k{}_j|$ which contradicts the definition of J.

For a smooth curve of complex structures J_t let $I = \frac{d}{dt} J_t|_{t=0} \in C^{\infty}(TX \otimes T^*X)$. Differentiating $-Id = J_t^2$ with respect to t shows that I satisfies $IJ + JI = 0$ and differentiating $N(J_t) = 0$ gives us $0 = N'_J(I) = \frac{1}{2} J \circ \bar{\partial} I$, where N is the Nijenhuis tensor of J. The tensor $g \circ I$ is skew-Hermitian as $g \circ I(JX, JY) = g(JX, IJY) = -g(JX, JIY) = -g \circ I(X, Y)$.

Definition 5.5.4. *Let (X, J, g) be a complex ALE$_\tau$ manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > m$. We define the space of* infinitesimal complex deformations $ICD^\infty_{-\tau}(J)$ *to be the subspace of $C^\infty_{-\tau}(TX \otimes T^*X)$ consisting of those I which satisfy $\bar\partial I = 0$ and $IJ + JI = 0$.*

We say that I is *symmetric* or *anti-symmetric* if $g \circ I$ is symmetric or anti-symmetric respectively. Denote by I^s and I^a the symmetric and anti-symmetric part of I respectively and by $ICD^\infty_{-\tau}(J)_S$ and $ICD^\infty_{-\tau}(J)_A$ the symmetric and anti-symmetric subspaces of $ICD^\infty_{-\tau}(J)$ respectively.

For a *Hermitian* symmetric tensor h_H the tensor $\psi = h_H \circ J$ satisfies $\psi(X,Y) = -\psi(Y,X)$, so it is a real differential 2-form of order $-\tau$. Also, $\mathbf{J}\psi(X,Y) = i^{1-1}\psi(X,Y)$, so $\psi = h_H \circ J$ is a differential form of type $(1,1)$. The same calculation as in (3.10) shows that for g Kähler,

$$\delta_g h_H = -(d^*\psi) \circ J. \tag{5.4}$$

Let ω be the Kähler form of g with respect to J. A local calculation shows that

$$\mathrm{tr}_g h_H = (\psi, \omega)_g. \tag{5.5}$$

By [BES87, (12.92')] The *Weitzenböck formula* for Hermitian tensors is

$$\Delta\psi = (\nabla^*\nabla - 2\overset{\circ}{R})h_H \circ J. \tag{5.6}$$

A *skew-Hermitian* symmetric tensor h_A can be identified with a real endomorphism $I \in End(TX)$ via

$$h_A \circ J = g \circ I. \tag{5.7}$$

The same calculation as in (3.14) shows that the endomorphism I from (5.7) satisfies the relation $g(X, IJY) = -g(X, JIY)$, so $g(X, (IJ + JI)Y) = 0$, i.e I anti-commutes with J. This implies, for $X \in T^{0,1}X$, that $J(IX) = -IJX = -I(-iX) = iIX$, so $IX \in T^{1,0}X$ and $I : T^{0,1}X \to T^{1,0}X$. I may therefore be regarded as an element of $\mathcal{A}^{0,1}(T^{1,0}X)$. The metric g is bounded but does not decay to zero and the form $h \circ J$ satisfies $h \circ J = O(r^{-\tau})$. Locally $(h_A)_{ik}J^k_j = g_{ik}I^k_j$. Hence each I^k_j satisfies $I^k_j = O(r^{-\tau})$. So we can think of I as an element of $\mathcal{A}^{0,1}_{-\tau}(T^{1,0}X)$. The tensor field $\bar\partial^* I \in C^\infty_{-\tau-1}(TX \otimes$

$T^*X)$ is related to $\delta_g h_A \in C^\infty_{-\tau-1}(T^*X)$ via

$$\delta_g h_A = -J \circ \bar{\partial}^* I. \tag{5.8}$$

In particular, $\delta_g h_A = 0$ exactly when I is $\bar{\partial}^*$-closed. From [BES87, (12.93')] we get a Weitzenböck formula for the complex Laplacian $\Delta_{\bar{\partial}}$,

$$(\nabla^*\nabla - 2 \overset{\circ}{R}) h_A \circ J = g \circ \Delta_{\bar{\partial}} I. \tag{5.9}$$

So $(\nabla^*\nabla - 2 \overset{\circ}{R}) h_A = 0$ exactly when $I \in \mathcal{A}^{0,1}_{-\tau}(T^{1,0}X)$ is in the kernel of $\Delta_{\bar{\partial}}$. For the convenience of the reader we restate Proposition 5.1.4.

Proposition 5.5.5. *Let (X, g) be an ALE$_\tau$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > m$. For an infinitesimal Ricci-flat deformation $h \in \epsilon^\infty_{-\tau}(g)$ both its Hermitian part and its skew-Hermitian part lie in $\epsilon^\infty_{-\tau}(g)$.*

Proof. Let $h = h_H + h_A$ be the decomposition of h into its Hermitian and skew-Hermitian parts. We show that $h_H \in \epsilon^\infty_{-\tau}(g)$. This suffices, as $\epsilon^\infty_{-\tau}(g)$ is a vector space, so $h_A = h - h_H \in \epsilon^\infty_{-\tau}(g)$. Expanding the definition of $\nabla^*\nabla - 2 \overset{\circ}{R}$ and using that on a Kähler manifold $\nabla_X JY = J\nabla_X Y$, a computation shows that $(\nabla^*\nabla - 2 \overset{\circ}{R}) h_A$ is anti-Hermitian and $(\nabla^*\nabla - 2 \overset{\circ}{R}) h_H$ is Hermitian. The decomposition of symmetric tensors into Hermitian and anti-Hermitian ones is orthogonal at all points and $\nabla^*\nabla - 2 \overset{\circ}{R}$ is linear. The tensor h satisfies $(\nabla^*\nabla - 2 \overset{\circ}{R}) h = 0$, which implies

$$(\nabla^*\nabla - 2 \overset{\circ}{R}) h_H = 0. \tag{5.10}$$

The tensor h_A is skew-Hermitian, so $\mathrm{tr}_g h_A = 0$. Hence $\mathrm{tr}_g h_H = \mathrm{tr}_g h - \mathrm{tr}_g h_A = 0$. To show that $\delta_g h_H = 0$ we use the Weitzenböck formula (5.6) to deduce from (5.10) that $\psi = h_H \circ J$ satisfies $\Delta\psi = 0$. The weight of ψ satisfies $-\tau < -m$ just as h_H, so by Lemma 5.3.5 ψ is d^*-closed. Equation (5.4) then tells us that $\delta_g h_H = 0$. Hence $h_H \in \epsilon^\infty_{-\tau}(g)$. □

Let (X, J, g) be a complex Kähler manifold. Let J_t be a smooth curve of complex structures through J and let g_t be a corresponding smooth curve of metrics through g such that each g_t is Kähler with respect to J_t. Let $h = \frac{d}{dt} g_t|_{t=0}$ and $I = \frac{d}{dt} J_t|_{t=0}$. The $(1,1)$-tensor $I = I^i_j$ is identified with a $(2,0)$-tensor $g \circ I$. The components of the latter tensor we denote I_{ij}. According to [KOI83, section 9] the tensors h and I_{ij} are related via

$$2ih_{\alpha\beta} + (I_{\alpha\beta} + I_{\beta\alpha}) = 0 \tag{5.11}$$

$$2i(D_\alpha h_{\beta\bar{\gamma}} - D_\beta h_{\alpha\bar{\gamma}}) = D_{\bar{\gamma}}(I_{\alpha\beta} - I_{\beta\alpha}), \tag{5.12}$$

where we use Greek indices as explained in the preliminaries. We say that a symmetric $(2,0)$-tensor h and an infinitesimal complex deformation I are *Kähler related* when they satisfy the relations (5.11) and (5.12).

The space $\epsilon^{\infty}_{-\tau}(g)$ can be written as $\epsilon^{\infty}_{-\tau}(g) = \epsilon^{\infty}_{-\tau}(g)_H \oplus \epsilon^{\infty}_{-\tau}(g)_A$, where $\epsilon^{\infty}_{-\tau}(g)_H$ and $\epsilon^{\infty}_{-\tau}(g)_A$ denote the subspaces of $\epsilon^{\infty}_{-\tau}(g)$ of Hermitian and skew-Hermitian tensor fields respectively. In the following two subsections the subspaces $\epsilon^{\infty}_{-\tau}(g)_H$ and $\epsilon^{\infty}_{-\tau}(g)_A$ will be studied separately.

Hermitian symmetric 2-tensors

Let (X, J, g) be an ALE$_\tau$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > m$. As explained above, any Hermitian symmetric $(2,0)$-tensor h can be identified with the $(2,0)$-tensor $h \circ J$ which is a real differential 2-form of type $(1,1)$. Let $h \in \epsilon^{\infty}_{-\tau}(g)_H$, then $(\nabla^*\nabla - 2\overset{\circ}{R})h = 0$ and the Weitzenböck formula, (5.6), gives us

$$\Delta(h \circ J) = (\nabla^*\nabla - 2\overset{\circ}{R})h \circ J = 0.$$

Proposition 5.3.1 guarantees that $\Delta(h \circ J) = 0$ implies $h \circ J \in \mathcal{H}^{1,1}_{-\tau,\mathbb{R}}(X)$. This way we have identified elements of $\epsilon^{\infty}_{-\tau}(g)_H$ with real harmonic $(1,1)$-forms with weight $-\tau$, i.e with element of the space $\mathcal{H}^{1,1}_{-\tau,\mathbb{R}}(X)$. Denote the Kähler form of g by ω. As outlined in (5.5) the trace of h with respect to g becomes $\mathrm{tr}_g(h) = (h \circ J, \omega)_g$. The tensor h is trace-free, so the form $h \circ J$ is orthogonal to ω.

Assume that $\tau = 2m - 1$. Then, by Theorem 5.3.2, we know that $\mathcal{H}^{1,1}_{-2m+1,\mathbb{R}}(X) \simeq \mathcal{H}^{1,1}_{\mathbb{R}}(X)$ The map $\epsilon^{\infty}_{-2m+1}(g) \to \mathcal{H}^{1,1}_{\mathbb{R}}(X) : h \mapsto h \circ J$ is linear. Locally the relation between h and the harmonic form $\psi = h \circ J$ is $\psi_{\alpha\bar{\beta}} = -ih_{\alpha\bar{\beta}}$, from which we see that the map is both injective and surjective. The map is therefore an isomorphism of vector spaces and we have

$$\epsilon^{\infty}_{-2m+1}(g)_H \simeq \mathcal{H}^{1,1}_{\mathbb{R}}(X)/\mathbb{R} \cdot \omega.$$

We have shown

Proposition 5.5.6. *Let (X, J, g) be an ALE$_{2m-1}$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $2m-1$. Then*

$$\epsilon^{\infty}_{-2m+1}(g)_H \simeq \mathcal{H}^{1,1}_{\mathbb{R}}(X)/\mathbb{R}\cdot\omega$$

is an isomorphism and $\dim\, \epsilon^{\infty}_{-2m+1}(g)_H = \dim H^{1,1}_{\mathbb{R}}(X) - 1$.

Remark 5.5.7. *[KOI83, Lemma 9.8] implies that the infinitesimal Ricci-flat deformation $h \in \epsilon^{\infty}_{-2m+1}(g)$ Kähler related to 0 are exactly the elements $h \in \epsilon^{\infty}_{-2m+1}(g)_H$.*

Skew-Hermitian symmetric 2-tensors

Let (X, J, g) be an ALE$_{2m-1}$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $2m-1$. From the above remarks about anti-Hermitian tensors $h_A \in \epsilon^{\infty}_{-2m+1}(g)_A$ we know that the endomorphisms I satisfying $h_A \circ J = g \circ I$ defines a $\bar{\partial}$-closed element $I \in \mathcal{A}^{0,1}_{-2m+1}(T^{1,0}X)$. The calculation in (3.19) is valid also on non-compact manifolds and it shows that this I satisfies $g \circ I(X, Y) = g \circ I(Y, X)$ so I is symmetric.

Lemma 5.5.8. *Let (X, J, g) be an ALE$_{2m-1}$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $2m-1$. The map*

$$\epsilon^{\infty}_{-2m+1}(g)_A \to \ker(\Delta_{\bar{\partial}}) \cap ICD^{\infty}_{-2m+1}(J)_S : h \mapsto I, \tag{5.13}$$

where I is given by the equation $h \circ J = g \circ I$, is bijective.

Proof. For $h \in \epsilon^{\infty}_{-2m+1}(g)_A$ the corresponding $I \in \mathcal{A}^{0,1}_{-2m+1}(T^{1,0}X)$ satisfies (5.8) and (5.9), which imply $\bar{\partial}^* I = 0$ and $\Delta_{\bar{\partial}} I = 0$ respectively. I has weight $-2m+1$, so by integration by parts,

$$0 = (0, I)_g = (\Delta_{\bar{\partial}} I, I)_g = (\bar{\partial}^* \bar{\partial} I, I)_g = ||\bar{\partial} I||^2_g.$$

Hence I is $\bar{\partial}$-closed and satisfies $I \in \ker(\Delta_{\bar{\partial}}) \cap ICD^{\infty}_{-2m+1}(J)_S$. The maps $h \mapsto h \circ J$ and $I_{ij} \mapsto I^i_j$ are injective, so $h \mapsto I$ is injective. To show surjectivity take an arbitrary $I \in \ker(\Delta_{\bar{\partial}}) \cap ICD^{\infty}_{-2m+1}(J)_S$. Observe that $h(X, Y) = -h \circ J(X, JY) = -g \circ I(X, JY) = g \circ (JI)(X, Y)$, so define a $h \in C^{\infty}_{-2m+1}(\mathrm{Sym}^2(T^*X))$ by $h(X, Y) = g \circ (JI)(X, Y)$.

This h satisfies $h \circ J = g \circ I$, i.e. h is mapped to I. By (5.9) the tensor h satisfies $(\nabla^*\nabla - 2 \overset{\circ}{R})h = 0$. The same calculation as in (3.20) shows that the tensor h is anti-Hermitian, i.e. $h(JX, JY) = -h(X, Y)$. This implies that $\mathrm{tr}_g h = 0$. The tensor h also satisfies (5.8), so $\delta_g h = 0$. Hence $(2\delta_g + d\,\mathrm{tr}_g)h = 0$ and so $h \in \epsilon^{\infty}_{-2m+1}(g)_A$. □

Remark 5.5.9. *The anti-symmetry of $I \in \mathcal{A}^{0,1}_{-2m+1}(T^{1,0}X)_A$ means that $g \circ I$ is a differential 2-form. The form $g \circ I$ is anti-Hermitian, so it is not of type $(1,1)$. I is a $(0,1)$-form with values in $T^{1,0}X$, so the form $g \circ I$ is naturally identified with a $(0,2)$-form. The same argument as in the proof of Corollary 3.7.9 shows that I is harmonic exactly if $g \circ I$ is harmonic. Hence elements of $\ker(\Delta_{\bar\partial}) \cap ICD^{\infty}_{-2m+1}(J)_A$ are in a one-to-one correspondence with $\mathcal{H}^{0,2}_{-2m+1}(X) \simeq H^{0,2}(X)$.*

Proposition 5.5.10. *Let (X, J, g) be an ALE$_{2m-1}$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $2m-1$. Then*

$$\dim \epsilon^{\infty}_{-2m+1}(g)_A = 2\dim_{\mathbb{C}} H^1(X, \mathcal{T}_X) - 2\dim_{\mathbb{C}} H^{0,2}(X, J).$$

Proof. Take $h_A \in \epsilon^{\infty}_{-2m+1}(g)_A$. From Lemma 5.5.8 it follows that the image of h_A in (5.13) is a $\bar\partial$-harmonic and symmetric element in $\mathcal{A}^{0,1}_{-2m+1}(T^{1,0}X)$ which defines a class $[I] \in H^1_{-2m+1}(X, \mathcal{T}_X)$. Harmonic anti-symmetric infinitesimal complex deformations correspond bijectively to harmonic $(0,2)$-forms on X via Remark 5.5.9. This completes the proof. □

Remark 5.5.11. *The relation $h \mapsto I$ from Lemma 5.5.8 can alternatively be written like $h_{\alpha\beta} = iI_{\alpha\beta}$, which is equivalent to saying that h is Kähler related to I. See [KOI83, Lemma 9.3] for details.*

5.6 Deformations of a pair $(\bar X, D)$

In this section, we introduce the deformation theory by Kawamata for a pair $(\bar X, D)$, where $\bar X$ is a compact complex manifold and D is a smooth divisor, and link it to the deformations of complex structures on $X = \bar X \backslash D$.

The section is divided into two parts. In the first part we introduce Kawamata's deformation theory and in the second part we link infinitesimal deformations of the pair $(\bar X, D)$ to infinitesimal complex deformations of $X = \bar X \backslash D$.

We are interested in studying complex deformations of X via the deformation theory for pairs $(\bar{X}, D)$ because we do not have a non-compact version of Kodaira's results about the stability of the Kähler property for small deformations of a complex structure over non-compact manifolds.

In the recent preprint [DER-KRO17] Deruelle-Kröncke proved an ALE version of the stability result of the Kähler property for complex deformations (more about this in Section 5.10). This could potentially have been used to simplify our work, but we decided not to follow their approach because it would have made a potential generalization of our results to asymptotically conical manifolds less natural (more about this in Section 5.11). We have for the same reason also formulated Kawamata's deformation theory in slightly more general terms than what would have been strictly needed for the context of ALE manifolds.

Deformations of the pair $(\bar{X}, D)$

Let $\bar{X}$ be a compact complex manifold and let D be a smooth divisor on $\bar{X}$. Following [KAW78] we introduce the deformation theory of the pair $(\bar{X}, D)$.

Definition 5.6.1. *Let $\bar{X}$ be a compact complex manifold with a smooth divisor D. We define the* sheaf of logarithmic p-forms $\Omega^p_{\bar{X}}(\log D)$ *to be the subsheaf of the sheaf of meromorphic p-forms consisting of those forms with a simple pole along D. The complex* $(\Omega^\bullet_{\bar{X}}(\log D), d)$ *is called the* holomorphic log complex.

A family of deformations of the pair $(\bar{X}, D)$ is defined as follows.

Definition 5.6.2. *Let $\bar{X}$ be a compact complex manifold and let D be a smooth divisor on $\bar{X}$. A* deformation of the pair $(\bar{X}, D)$ *is a tuple $(\bar{\mathcal{X}}, \mathcal{X}, \mathcal{D}, \bar{\pi}, S, s_0, \psi)$, where $\bar{\mathcal{X}}$ and S are complex spaces and $\bar{\pi} : \bar{\mathcal{X}} \to S$ is a proper smooth morphism. Fix $s_0 \in S$ and assume that the tuple satisfies*

i) $\psi : \bar{X} \to \bar{\pi}^{-1}(s_0)$ is an isomorphism of complex manifolds satisfying $\psi(X) = \bar{\pi}^{-1}(s_0) \cap \mathcal{X}$, where $X = \bar{X} \backslash D$.

ii) $\mathcal{D}$ is a closed analytic subset of $\bar{\mathcal{X}}$ and $\mathcal{X} = \bar{\mathcal{X}} - \mathcal{D}$.

iii) $\bar{\pi}$ is locally a projection of a product space, that is, for each $p \in \bar{\mathcal{X}}$ there exists an open neighbourhood U_p of p in $\bar{\mathcal{X}}$ and an isomorphism $\phi_p : U_p \simeq \bar{\pi}(U_p) \times W$, where

$W = U_p \cap \bar{\pi}^{-1}(\bar{\pi}(p))$ *such that the diagram*

$$\begin{array}{ccc} U_p & \xrightarrow{\phi_p} & \bar{\pi}(U_p) \times W \\ & {\scriptstyle \bar{\pi}} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & \bar{\pi}(U_p) & \end{array}$$

commutes.

iv) $\bar{\pi}$ *is the restriction of a product space to* $\mathcal{D}$*, i.e. for* $p \in \bar{\mathcal{X}}$ *we have* $\phi_p(U_p \cap \mathcal{D}) = \bar{\pi}(U_p) \times (W \cap \mathcal{D})$.

For each $t \in S$ in Definition 5.6.2 the pre-image $\bar{\pi}^{-1}(t)$ is a compact complex submanifold of $\bar{\mathcal{X}}$ diffeomorphic to $\bar{X}$ and $\bar{\pi}^{-1}(t) \cap \mathcal{D}$ is a smooth divisor on X_t diffeomorphic to D. In this case the family of deformations of the pair $(\bar{X}, D)$ produces in particular a family of complex deformations $\bar{X}_t$ of X and a family complex deformation D_t of D for $t \in S$.

Definition 5.6.3. *For a pair* $(\bar{X}, D)$ *define the logarithmic tangent sheaf* $\mathcal{T}_{\bar{X}}(\log D)$ *to be the subsheaf of the sheaf of holomorphic sections* $\mathcal{T}_{\bar{X}}$ *consisting of derivations of* $\mathcal{O}_{\bar{X}}$ *which sends the ideal sheaf of D,* $\mathcal{O}_D$*, into itself.*

Definition 5.6.4. *For a pair* $(\bar{X}, D)$*,* $\mathcal{T}_{\bar{X}}(\log D)$ *is the sheaf of infinitesimal automorphisms of* $\bar{X}$ *which send* D *into itself. For a pair* $(\bar{X}, D)$ *let* S *be the parameter space from Definition 5.6.2. By [KAW78, p. 249] there is a* Kodaira-Spencer map

$$\rho_{s_0} : T_{s_0}S \to H^1(\bar{X}, \mathcal{T}_{\bar{X}}(\log D)),$$

and $H^1(\bar{X}, \mathcal{T}_{\bar{X}}(\log D))$ *is the space of* infinitesimal deformations *of the pair* $(\bar{X}, D)$.

It follows from [KAW78, Theorem 1] that deformations of a pair $(\bar{X}, D)$ always exists and it follows from [KAW78, Corollary 4] that when $H^2(\bar{X}, \mathcal{T}_{\bar{X}}(\log D)) = 0$, then the parameter space S is regular at s_0 and each infinitesimal deformation integrates into a smooth curve of deformations $(\bar{X}_t, D_t)$.

Linking $\epsilon^{\infty}_{-2m+1}(g)_A$ to infinitesimal deformations of $(\bar{X}, D)$

Let $\bar{X}$ be a compact complex manifold and let D be a smooth divisor. The Riemann extension theorem tells us when a holomorphic function on $\bar{X} \backslash D$ extends across the divisor

D.

Theorem 5.6.5. *Let f_1 be a non-zero holomorphic function on a polydisc $\Delta \subset \mathbb{C}^m$ and let f_2 be a bounded and holomorphic function on $\bar{\Delta}\backslash\{f_1 = 0\}$. Then f_2 extends uniquely to a holomorphic function $\bar{f}_2$ on all of the polydisc Δ. ([GUN-ROS65, Theorem C.3]).*

For the pair $(\bar{X}, D)$ we can always extend a bounded holomorphic p-form η on X to $\bar{X}$, by extending the coefficients of η_U in each chart (U, ϕ) from $U \cap X$ to U using Theorem 5.6.5.

Lemma 5.6.6. *Let (X, J, g) be a complex ALE$_\tau$ manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > 0$ and assume that X arises as the complement of a smooth divisor D in a compact complex manifold $(\bar{X}, \bar{J})$. For $I \in \mathcal{A}^{0,1}_{-\tau}(T^{1,0}X)$ there exists $\bar{I} \in \mathcal{A}^{0,1}(T^{1,0}\bar{X})$ satisfying $\bar{I}|_X = I$. If $\bar{\partial} I = 0$, then $\bar{\partial}\bar{I} = 0$. Also, any $\bar{I} \in \mathcal{A}^{0,1}_{-\tau}(T^{1,0}X)$ is in particular an element of $\mathcal{A}^{0,1}(\mathcal{T}_{\bar{X}}(\log D))$.*

Proof. Outside a compact set the form $I \in \mathcal{A}^{0,1}_{-\tau}(T^{1,0}X)$ decays to zero, so it is bounded. In particular, all coefficients of I in the asymptotic coordinates π are holomorphic and bounded. By Theorem 5.6.5 all coefficients of I extend uniquely across D to holomorphic functions on all of $\bar{X}$. Hence there exists an $\bar{I} \in \mathcal{A}^{0,1}(T^{1,0}\bar{X})$ satisfying $\bar{I}|_X = I$. If $I \in \mathcal{A}^{0,1}(T^{1,0}X)$ satisfies $\bar{\partial} I = 0$, then the extended form $\bar{I}$ satisfies $\bar{\partial}\bar{I} = 0$ on X and the same argument as before shows that $\bar{\partial}\bar{I} = 0$ on all of $\bar{X}$. □

Lemma 5.6.7. *Assume the hypotheses of Lemma 5.6.6 and Lemma 5.5.8. Then each $h \in \epsilon^{\infty}_{-2m+1}(g)_A$ corresponds to an infinitesimal deformation $[\bar{I}] \in H^1(\bar{X}, \mathcal{T}_{\bar{X}}(\log D))$ of the pair $(\bar{X}, D)$.*

5.7 Stability results for deformations of a pair $(\bar{X}, D)$

In this section, we prove stability results for deformations of the pair $(\bar{X}, D)$ for a compact Kähler manifold $\bar{X}$ and a smooth divisor D.

Lemma 5.7.1. *Let $\bar{X}$ be a compact Kähler manifold and let D be a divisor biholomorphic to $\mathbb{CP}^{m-1}$. Assume that the pair $(\bar{X}, D)$ admits a family of deformations $(\bar{X}_t, D_t)$. Then for small t the divisor D_t is biholomorphic to $\mathbb{CP}^{m-1}$.*

Proof. A deformation $(\bar{X}_t, D_t)$ of the pair $(\bar{X}, D)$ is in particular a complex deformation of D, so for small t, D_t is Kähler. For small t, D_t is diffeomorphic to $D \simeq \mathbb{CP}^{m-1}$. It follows from [BES87, Theorem 11.30] that any Kähler manifold homeomorphic to $\mathbb{CP}^{m-1}$ is biholomorphic to $\mathbb{CP}^{m-1}$. In particular, D_t is biholomorphic to $\mathbb{CP}^{m-1}$. □

Lemma 5.7.2. *Let $\bar{X}$ be a compact Kähler manifold and let D be a smooth divisor. Assume that $(\bar{X}, D)$ admits a smooth family of deformation $(\bar{X}_t, D_t)$ and assume that the line bundle L_D is ample. Then for small t, L_{D_t} is ample.*

Proof. This is a direct consequence of the Kodaira Embedding Theorem. It says that a line bundle on a compact Kähler manifold $(\bar{X}, J)$ is positive if and only if it is ample. By definition a line bundle is positive if the first Chern class can be represented by a positive real $(1,1)$-form, which is equivalent to the positive definiteness of the corresponding symmetric $(2,0)$-tensor, which is an open condition. □

Proposition 5.7.3. *Let $\bar{X}$ be a compact complex manifold with a smooth divisor D satisfying $K_{\bar{X}} = -\beta L_D$, i.e. $c_1(K_{\bar{X}}) = -\beta c_1(L_D)$, for some $\beta \geq 1$. Let $(\bar{X}_t, D_t)$ be a smooth family of deformations of the pair $(\bar{X}, D)$. Then for small t, $K_{\bar{X}_t} = -\beta L_{D_t}$.*

Proof. The first Chern class of the line bundles $K_{\bar{X}_t}$ and L_{D_t} take values in $H^2(\bar{X}, \mathbb{Z})$ and are therefore stable for small deformations of the pair $(\bar{X}, D)$, i.e. $c_1(K_{\bar{X}_t}) = c_1(K_{\bar{X}})$ and $c_1(L_{D_t}) = c_1(L_D)$ for small t. In particular, $c_1(K_{\bar{X}}) = -\beta c_1(L_D)$ implies that $c_1(K_{\bar{X}_t}) = -\beta c_1(L_{D_t})$ for small t. □

Proposition 5.7.4. *Let $\bar{X}$ be a compact Kähler manifold with a smooth ample divisor D satisfying $K_{\bar{X}} = -\beta L_D$ for some $\beta \geq 1$ and assume that $X = \bar{X} \backslash D$ admits an ALE$_{2m}$ Ricci-flat Kähler metric g. Also assume that the pair $(\bar{X}, D)$ admits a smooth family of deformations $(\bar{X}_t, D_t)$. Then there exists a smooth family of complete Kähler metrics g_t for the family $X_t = \bar{X}_t \backslash D_t$ through g.*

Proof. For the smooth family of deformations $(\bar{X}_t, D_t)$ of the pair $(\bar{X}, D)$ denote by $\bar{J}_t$ the corresponding family of complex structures on $\bar{X}_t$ and by J_t the restriction of $\bar{J}_t$ to X_t. It satisfies $\bar{J}_0 = \bar{J}$, where $\bar{J}$ is the complex structure on $\bar{X}$. From Proposition 5.7.3 it follows that $c_1(K^{-1}_{\bar{X}_t} \otimes \beta L^{-1}_{D_t}) = 0$.

For $\beta = 1$ [TIA-YAU90, Theorem 4.1] ensures that for every $\rho_t \in c_1(K^{-1}_{\bar{X}_t} \otimes L^{-1}_{D_t}) = 0$ there exists a complete Kähler metric g_t on X_t compatible with J_t and with Ricci-form ρ_t.

For $\beta > 1$ [TIA-YAU91, Theorem 1.1] ensures that for every $\rho_t \in c_1(K_{\bar{X}_t}^{-1} \otimes L_{D_t}^{-1}) = 0$ there exists a complete Kähler metric g_t on X_t compatible with J_t and with Ricci-form ρ_t.

By the explicit construction of the metric g_t both in [TIA-YAU90, Theorem 4.1] and [TIA-YAU91, Theorem 1.1] and since g can be taken to arise as a result of these constructions, then there exists a smooth curve of complete Kähler metrics g_t with respect to J_t and with Ricci-forms ρ_t passing through g. □

We feel that an ALE version of the Kähler stability result [KOD-SPE60, Theorem 15] which does not make use of a compactification would have been more natural than Proposition 5.7.4. This was recently achieved by Deruelle and Kröncke in the preprint [DER-KRO17]. To keep our work independent of [DER-KRO17] we have decided not to make use of their findings. See Section 5.10 for a discussion of [DER-KRO17].

5.8 Proof of Theorem 5.1.5

In this section, we prove Theorem 5.1.5.

proof of Theorem 5.1.5. Let (X, J, g) be an ALE_{2m} Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$. Assume that $X = \bar{X}\backslash D$ for a compact complex Kähler manifold $(\bar{X}, \bar{J}, \bar{g})$ with a smooth ample divisor D satisfying $K_{\bar{X}} = -\beta L_D$ for some $\beta \geq 1$. Also assume that all infinitesimal deformations of the pair $(\bar{X}, D)$ are integrable.

Denote by $\mathcal{J}$ the parameter space for the family of deformations of the pair $(\bar{X}, D)$. The assumption that all infinitesimal complex deformations are integrable implies that the space $\mathcal{J}$ is smooth in a neighbourhood around J. Denote by $\mathcal{V}$ the vector bundle over $\mathcal{J}$ with fibres $\mathcal{V}_t = \mathcal{H}^{1,1}_{-2m+1,\mathbb{R}}(X, J_t)$ of harmonic real 2-forms with weight $-2m+1$ and of type $(1,1)$ with respect to J_t.

Take $h \in \epsilon^{\infty}_{-2m+1}(g)$. It is Kähler related to some $I \in \mathcal{A}^{0,1}_{-2m+1}(T^{1,0}X)$ which by Lemma 5.6.7 gives rise to an infinitesimal deformation $[\bar{I}] \in H^1(\bar{X}, \mathcal{T}_{\bar{X}}(\log D))$ of the pair $(\bar{X}, D)$. Infinitesimal deformations of the pair $(\bar{X}, D)$ are assumed to be integrable. For a smooth curve of deformations $(\bar{X}_t, D_t)$ let $X_t = \bar{X}_t\backslash D_t$ and denote the corresponding family of complex structures on $\bar{X}_t$ by $\bar{J}_t$ and the restriction to X_t by J_t. Write $J_t = J + I_t$. Choose a smooth curve of deformations $(\bar{X}_t, D_t)$ of the pair $(\bar{X}, D)$ which satisfies $\frac{d}{dt}J_t|_{t=0} = \frac{d}{dt}I_t|_{t=0} = I$. For small t Proposition 5.7.3 says that $K_{\bar{X}_t} = -\beta L_{D_t}$ and Lemma 5.7.2 says that L_{D_t} is ample, so by Proposition 5.7.4 there exists a smooth curve of complete Kähler metrics g_t on X_t passing through $g_0 = g$. Write $g_t = g + h_t$. Each metric g_t is Kähler with respect to J_t so $\frac{d}{dt} g_t|_{t=0} = g_0' = h_0'$ and $\frac{d}{dt} J_t|_{t=0} = I_0' = I$ are Kähler related (as explained in Section 5.5). The tensor I has weight $-2m+1$, so g_0' also has weight $-2m+1$, and then so does each h_t. The Kähler metrics g_t are therefore ALE_{2m-1}. Define the form $\kappa = (\frac{d}{dt} g_t|_{t=0} - h) \circ J$. It is a real $(1,1)$-form and by (5.11) and (5.12) it is closed and Hermitian. Take a curve κ_t in $\mathcal{V}$ of Hermitian forms such that $\kappa_0 = \kappa$. Denote by ω_t the Kähler form $g_t \circ J_t$. The form $\tilde{\omega}_t = \omega_t - t\kappa_t$ is closed and the tensor $\tilde{g}_t = (\omega_t - t\kappa_t) \circ J_t$ is positive definite, so $\tilde{g}_t$ is a Kähler metric with respect to J_t. The symmetric tensor $t\kappa_t \circ J_t$ has weight $-2m+1$ and the metric $\tilde{g}_t$ is ALE_{2m-1}. By Theorem 5.3.6 the real $(1,1)$-forms cohomologous to the Kähler form $\tilde{omega}_t = \omega_t - t\kappa_t$ in $H^2_{-2m+1}(X, \mathbb{R})$ are parametrized by $u \in C^\infty_{-2m+3}(X)$ via $\omega_t - t\kappa_t + i\partial\bar{\partial}u$. Solutions $u \in C^{k+2,\alpha}_{-2m+3}(X)$ to the complex Monge-Ampère equation

$$(\omega_t - t\kappa_t + i\partial\bar{\partial}u)^m + e^f(\omega_t - t\kappa_t)^m = 0 \tag{5.14}$$

produces Kähler forms $\hat{\omega}_t = \omega_t - t\kappa_t + i\partial\bar{\partial}u$. Note that for $t = 0$ the function $u = 0$ produces the already known ALE_{2m} Ricci-flat Kähler metric g on (X, J). For each t the Ricci-form $\tilde{\rho}_t$ of g_t belongs to $2\pi c_1(J_t) = 0$, so there exists a smooth curve of functions $f_t \in C^\infty(V)$ satisfying $\tilde{\rho}_t = i\partial\bar{\partial} f_t$. Similarly to what was done in Section 3.5 we can isolate f in (5.14) to produces a function $F : \mathcal{V} \times C^{5,\alpha}_{-2m+3}(X) \to C^{3,\alpha}_{-2m+1}(X)$ which in holomorphic coordinates $\{z^1, \dots, z^m\}$ is given by

$$F((t, \kappa_t), u) = \log\det\left(\tilde{g}_{\alpha\bar{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}\right) - \log\det(\tilde{g}_{\alpha\bar{\beta}}) - f_t.$$

A similar calculation to (3.3) shows that the linearization of F in u is

$$\frac{d}{ds} F((t, \kappa_t), u + sv)|_{s=0} = \Delta_{\hat{g}} v,$$

where $\hat{g}$ is the ALE_{2m-1} Kähler metric with Kähler form $\hat{\omega} = \tilde{\omega} + i\partial\bar{\partial}u$. For each $v' \in C^{3,\alpha}_{-2m+1}(X)$ we can by [JOY00, Theorem 8.3.6] find a unique solution $v \in C^{5,\alpha}_{-2m+3}(X)$ such that $\Delta_{\hat{g}} v = v'$. The linearization of F is therefore surjective and F is a submersion. From the implicit function theorem for Banach spaces (Theorem 2.1.10) we conclude that there exists a smooth function $\psi : \mathcal{U} \subset \mathcal{V} \to C^{5,\alpha}_{-2m+3}(X)$ from a neighbourhood $\mathcal{U}$

of $(0,0)$ in $\mathcal{V}$ to a neighbourhood of 0 in $C^{5,\alpha}_{-2m+3}(X)$ such that for each $(J_t,\kappa_t)\in\mathcal{U}$, ψ produces an ALE_{2m-1} Ricci-flat Kähler metric $\hat{g}_t=\psi(J_t,\kappa_t)$, where $\hat{\omega}_t=\omega_t-t\kappa_t+i\partial\bar{\partial}u$ and $\hat{g}_t$ is the Ricci-flat Kähler metric satisfying $\hat{g}_t\circ J_t=\hat{\omega}_t$. It follows from [DeT-KAZ81, Theorem 6.1] that the metric $\hat{g}_t$ is actually smooth. Restrict ψ to a subset $\tilde{\mathcal{U}}$ of $\mathcal{U}$ such that $\psi|_{\tilde{\mathcal{U}}}$ only takes values in the slice S_{2m-1}. The image $\psi(\tilde{\mathcal{U}})$ is contained in the premoduli space P_g and $d\psi(T_{(0,0)}\tilde{\mathcal{U}})$ is contained in $\epsilon^{\infty}_{-2m+1}(g)$.

The rest of the proof consists in showing that ψ surjects onto a a neighbourhood of g in the slice S_{2m-1}. This part of the proof is similar to the latter part of the proof of Theorem 3.1.3. We repeat it for the convenience of the reader but with fewer details. Take the $h\in\epsilon^{\infty}_{-2m+1}(g)$ Kähler related to $I=\frac{d}{dt}J_t|_{t=0}$ from before. For the Kähler form $\tilde{\omega}_t=\omega_t-t\kappa_t$ the form $\tilde{\omega}_0'-h\circ J$ satisfies

$$\tilde{\omega}_0'-h\circ J=\frac{d}{dt}(g_t-t\phi_t)|_{t=0}\circ J+\tilde{g}\circ I-h\circ J=\tilde{g}\circ I^a+\tilde{g}\circ I^s,$$

where ϕ_t denotes the tensor satisfying $\kappa_t=\phi_t\circ J_t$. The tensor $\tilde{g}\circ I^s$ is symmetric and the tensor $\tilde{g}\circ I^a$ satisfies $\mathbf{J}(\tilde{g}\circ I^a)(X,Y)=-\tilde{g}\circ I^a(X,Y)$, so the 2-form $\tilde{g}\circ I^a$ is not of type $(1,1)$. Hence $[\tilde{\omega}_0'-h\circ J]=[\tilde{g}\circ I]=0$ in $H^{1,1}_{-2m+1}(X,J)$. Now,

$$\hat{\omega}_0'=\frac{d}{dt}(\tilde{\omega}_t+i\partial\bar{\partial}u_t)|_{t=0}=\tilde{\omega}_0'+i\partial\bar{\partial}u_0'$$

so $[\hat{\omega}_0']=[\tilde{\omega}_0']$. Hence $[\hat{\omega}_0'-h\circ J]=0$.

The symmetric tensors $\hat{g}_0'$ and h are both Kähler related to $I=J_0'$, so $\hat{g}_0'-h$ is Kähler related to 0 and it is therefore Hermitian by Remark 5.5.7. For the Hermitian tensor $\hat{g}_0'-h\in\epsilon^{\infty}_{-2m+1}(g)$ the form $(\hat{g}_0'-h)\circ J$ is harmonic by the Weitzenböck formula for Hermitian tensors (5.6), since

$$\Delta((\hat{g}_0'-h)\circ J)=(\nabla^*\nabla-2\overset{\circ}{R})(\hat{g}_0'-h)\circ J=0.$$

Now,

$$\begin{aligned}0=[\hat{\omega}_0'-h\circ J]=[(\hat{g}_0'-h)\circ J+\hat{g}\circ I]&=[(\hat{g}_0'-h)\circ J]+[\hat{g}\circ I]\\&=[(\hat{g}_0'-h)\circ J].\end{aligned}$$

A harmonic form cohomologous to zero is itself zero, so $(\hat{g}_0'-h)\circ J=0$, which implies that $\hat{g}_0' = h$. Hence the differential of the map ψ is surjective and ψ surjects onto a neighbourhood of g in S_{2m-1}. This concludes the proof. □

5.9 Moduli space of ALE Ricci-flat structures

In this section, we introduce the moduli space of ALE Ricci-flat structures and prove that in a neighbourhood of an ALE Ricci-flat Kähler structure it is, up to an action of a finite group, a finite dimensional manifold and we find an expression for its dimension using cohomology groups. This is the content of Theorem 5.1.6, which is an ALE version of Theorem 2.7.2. The moduli space constructed in this section is different from the one used in Section 2.7 and Chapter 3, reflecting the fact that we no longer work over a compact base.

Let (X, J, g) be an ALE$_\tau$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ of order $\tau > 0$. In Section 5.4 we denoted by $\mathcal{M}$ the space of complete Riemannian metrics and by $\mathcal{M}_{ALE_\tau}$ the subspace of $\mathcal{M}$ consisting of ALE$_\tau$ metrics. In Section 5.4 we showed that the group of diffeomorphism $\mathcal{D}_{\tau+1}$ acts on $\mathcal{M}_{ALE_\tau}$. We introduce the equivalence relation $\sim$ on $\mathcal{M}$. Two metrics $g, g' \in \mathcal{M}_{ALE_\tau}$ are equivalent if there exists $\phi \in \mathcal{D}_{\tau+1}$ such that $g' = \phi^* g$. The equivalence class of an ALE$_\tau$ metric is called an *ALE$_\tau$ structure*. The quotient space $\widetilde{\mathcal{M}}_{ALE_\tau} = \mathcal{M}_{ALE_\tau}/\sim$ is equipped with the quotient topology.

Denote by $\mathcal{R} \subset \mathcal{M}$ the complete Ricci-flat metrics and by $\mathcal{R}_{ALE_\tau} \subset \mathcal{M}_{ALE_\tau}$ the ALE_τ Ricci-flat metrics. Denote by $\widetilde{\mathcal{R}}_{ALE_\tau}$ the subspace of $\widetilde{\mathcal{M}}_{ALE_\tau}$ consisting of ALE$_\tau$ structures containing a Ricci-flat metric. We call such structures for ALE$_\tau$ Ricci-flat structures and call $\widetilde{R}_{ALE_\tau}$ for the *moduli space* of ALE$_\tau$ Ricci-flat structures. The pull-back of a Ricci-flat metric by a diffeomorphism is Ricci-flat, so any metric in a Ricci-flat structure is Ricci-flat. We can therefore equivalently express $\widetilde{\mathcal{R}}_{ALE_\tau}$ as $\mathcal{R}_{ALE_\tau}/\sim$ or $\mathcal{R}_{ALE_\tau}/\mathcal{D}_{\tau+1}$.

Lemma 5.9.1. *Let (X, g) be an ALE$_\tau$ manifold asymptotic to $\mathbb{R}^n/G$ of order $\tau > 0$ and assume that g is not flat. Then the isometry group Iso(X, g) is a finite dimensional compact Lie group.*

Proof. By [BES87, Theorem 1.77] the group Iso(X, g) is a Lie group. A metric space is compact if every sequence has a convergent subsequence with a limit point in the space. From [KOV06, proof of Lemma 3.6] we know that a sequence $\{\eta_i\}_{i\in\mathbb{Z}_{\geq 0}}$ in Iso(X, g) has a convergent subsequence if for some $p \in X$ the sequence $\{\eta_i(p)\}_{i\in\mathbb{Z}_{\geq 0}}$ is convergent. The metric g is not flat so we can take some $p_0 \in X$ with positive curvature, say $|R_{p_0}|_g > \epsilon > 0$. On ALE manifolds the Riemann curvature tensor decays to zero, so $|R_p|_g < \epsilon$ for $\rho(p) > C$ for some $C > 0$, where ρ is the radius function on X. Hence $\eta(p_0)$ is contained in the compact ball of radius C for any isometry η. Hence the sequence $\{\eta_i(p_0)\}_{i\in\mathbb{Z}_{\geq 0}}$ has a limit point in X and the sequence $\{\eta_i\}_{i\in\mathbb{Z}_{\geq 0}}$ has a convergent subsequence. Iso(X, g) is therefore compact. □

Denote by I_g^0 the connected component of the identity element in the Lie group of isometries Iso(X, g). It is called the *identity component* and it is a normal subgroup. When the Lie group Iso(X, g) is compact the quotient Iso$(X, g)/I_g^0$ is a finite group.

Lemma 5.9.2. *Let (X, g) be an ALE$_{n-1}$ Ricci-flat manifold asymptotic to $\mathbb{R}^n/G$. The identity component I_g^0 acts trivially on P_g.*

Proof. Let $g' \in S_{n-1}$. Then $\eta \in$ Iso(X, g') fixes g' in S_{n-1} so Theorem 5.4.7(2) implies that Iso$(X, g') \subset$ Iso(X, g). In particular $I_{g'}^0 \subset I_g^0$. Let $g' \in \mathcal{R}_{ALE_{n-1}}$. On (X, g') let $\alpha \in C^\infty_{-n+1}(\wedge^1 T^*X)$ be a harmonic 1-form. As g' is Ricci-flat the Weitzenböck formula ensures that $\Delta_{g'}\alpha = \nabla^*_{g'}\nabla_{g'}\alpha$ and integration by parts then gives us $\nabla_{g'}\alpha = 0$. For 1-forms $\delta^*_{g'}\alpha = \nabla\alpha - \frac{1}{2}d\alpha = 0 + 0 = 0$. Hence $\mathcal{L}_{\alpha^\sharp} g = \delta^*_{g'}\alpha = 0$, so $\alpha^\sharp$ is a Killing vector field. On the other hand, Let X be a Killing vector field, then $\delta^*_{g'}X^\flat = \mathcal{L}_X g = 0$, and from Proposition 5.4.5 we get $\Delta_{g'}X^\flat = (2\delta_{g'} + d\,\mathrm{tr}_g)\delta^*_{g'}X^\flat = 0$, so $X^\flat$ is harmonic. Killing fields are therefore in bijection with harmonic 1-forms. The Laplace operator is Fredholm, so the space of harmonic 1-forms is finite dimensional. For any $g' \in P_g$ the connected Lie subgroup $I_{g'}^0$ therefore coincides with I_g^0. □

Proof of Theorem 5.1.6. Let (X, J, g) be an ALE$_{2m}$ Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ satisfying the assumptions of Theorem 5.1.5. From Section 5.4 we know that there exists a slice $S_{2m-1} \subset \mathcal{M}_{ALE_{2m-1}}$ for the action of $\mathcal{D}_{2m}$. From Theorem 5.5.3 we know that there exists a finite dimensional manifold $Z \subset S_{2m-1}$ with tangent space $T_g Z = \epsilon^\infty_{-2m+1}(g)$ at g and with the premoduli space of Ricci-flat metrics P_g as a real analytic subset. From Theorem 5.1.5 we know that all infinitesimal Ricci-flat deformations integrate into Ricci-flat deformations, so P_g is itself a manifold with $T_g P_g = \epsilon^\infty_{-2m+1}(g)$. As a result of the slice construction we know that a neighbourhood $U \subset \widetilde{\mathcal{R}}_{ALE_{2m-1}}$ of $[g]$ is homeomorphic to $P_g/$Iso(X, g). By Lemma 5.9.1 and Lemma 5.9.2 it is homeomorphic to the orbifold $P_g/($Iso$(X, g)/I_g^0)$. The dimension of $\widetilde{\mathcal{R}}_{ALE_{2m-1}}$ at $[g]$ is therefore the dimension of P_g at g. We can give an expression for this dimension via Proposition 5.5.6 and Proposition 5.5.10.

$$\begin{aligned}
&\dim T_g P_g \\
&\quad = \dim \epsilon^{\infty}_{-2m+1}(g) \\
&\quad = \dim \epsilon^{\infty}_{-2m+1}(g)_H + \dim \epsilon^{\infty}_{-2m+1}(g)_A \\
&\quad = \dim \mathcal{H}^{1,1}_{\mathbb{R}}(X,J) - 1 + 2\dim_{\mathbb{C}} H^1(X,\mathcal{T}_X) - 2\dim_{\mathbb{C}} H^{0,2}(X,J).
\end{aligned}$$

Since $H^{0,2}(X,J) = 0$ by [JOY00, Theorem 8.4.3], this concludes the proof. $\square$

5.10 ALE Ricci-flat deformations revisited

In this section, we discuss a recent preprint ([DER-KRO17]), where Deruelle and Kröncke prove a similar result to Theorem 5.1.5. Their work and the work in this thesis have been done independently. In this section we briefly summarize their findings and highlight a few differences between their approach and ours. Their result ([DER-KRO17, Theorem 2.17]) is

Theorem 5.10.1. *Let (X,J,g) be an ALE Calabi-Yau manifold asymptotic to $\mathbb{C}^m/G$ and let $\delta \in (-2m+1,-2m+2)$ be not exceptional. Then for any $h \in \ker_{L^2}(\Delta_L)$, there exists a smooth family g_t of Ricci-flat metrics in some slice $S \subset \mathcal{M}^2_{k,\delta}$ through g for the action of $\mathcal{D}^2_{k+1,\delta+1}$, with $g_0 = g$ and $\frac{d}{dt}g_t|_{t=0} = h$. Each metric g_t is ALE and Kähler with respect to some complex structure J_t which is $L^2_{k,\delta}$-close to J.*

Here $\mathcal{D}^2_{k+1,\delta+1}$ denotes diffeomorphisms from X to itself which are $L^2_{k+1,\delta+1}$-close to the identity. The space $\ker_{L^2}(\Delta_L)$ is defined as $\ker(\Delta_L) \cap L^2(\mathrm{Sym}^2(T^*X))$, where Δ_L is the Lichnerowicz Laplacian $\Delta_L : C^\infty(\mathrm{Sym}^2(T^*X)) \to C^\infty(\mathrm{Sym}^2(T^*X))$. Any $h \in \ker_{L_2}(\Delta_L)$ can be shown to satisfy $h = O(\rho^{-2m-1})$ for a radius function ρ on the ALE manifold (X,g) of complex dimension m. $\mathcal{M}^2_{k,\delta}$ is the space of metrics $L^2_{k,\delta}$-close to g.

In the proof of Theorem 5.1.5 we made use of a compactification $X = \bar{X}\backslash D$ and deformation theory of the pair $(\bar{X}, D)$ to deform the complex structure in a way that preserved the Kähler property. Deruelle-Kröncke use a different approach. Instead of using a compactification they adapt the original proof ([KOD-SPE60, Theorem 15]) by Kodaira-Spencer to the ALE setting and show that the Kähler property is preserved under small deformations of the complex structure. This is the content of [DER-KRO17, Theorem 2.15] which we state next. Denote by $\mathcal{J}^2_{k,\delta}$ the space $L^2_{k,\delta}(\wedge^{0,1}X \otimes T^{1,0}X) \cap \ker(\Delta)$.

Theorem 5.10.2. *Let (X, J, g) be an ALE Ricci-flat Kähler manifold asymptotic to $\mathbb{C}^m/G$ and let $\delta < -2m + 2$ be not exceptional and $\mathcal{J}^2_{k,\delta}$ defined as above. Then there exists a $L^2_{k,\delta}$-neighbourhood U of J in $\mathcal{J}^2_{k,\delta}$ and a smooth map $\Psi : U \to \mathcal{M}^2_{k,\delta}$ which associates to each $J_t \in U \subset \mathcal{J}^2_{k,\delta}$ sufficiently close to $J_0 = J$ a metric g_t which is $L^2_{k,\delta}$-close to g_0 and Kähler with respect to J_t.*

While we made use of the auxiliary hypothesis that all infinitesimal deformations of the pair $(\bar{X}, D)$ are integrable, then Kröncke-Deruelle show that infinitesimal complex deformations of ALE Calabi-Yau manifolds always are integrable. This is the content of [DER-KRO17, Theorem 2.14].

Theorem 5.10.3. *Let (X, J, g) be an ALE Kähler manifold asymptotic to $\mathbb{C}^m/G$ with a holomorphic volume form. Let $k > m + 1$, $\delta < -m$ not exceptional and $I \in L^2_{k,\delta}(\wedge^{0,1} \otimes T^{1,0}X)$ such that $\bar{\partial}I = 0$ and $\bar{\partial}^*I = 0$. Then there exists a smooth family of complex structures J_t with $J_0 = J$ such that $J_t - J \in L^2_{k,\delta}(T^*X \otimes TX)$ and $\frac{d}{dt}J_t|_{t=0} = I$.*

There are a number of similarities between both statement and proof of [DER-KRO17, Theorem 2.17] and Theorem 5.1.5, but there are also differencies. Our approach is closer in spirit to the constructions in [CAL79], [BAN-KOB88], [BAN-KOB90], [TIA-YAU90] and [TIA-YAU91], where a defining section of the divisor is used to construct a complete metric on the complement of the divisor. An advantage of the approach by Deruelle-Kröncke is that they do not make use of a compactification. This would have been an attractive simplification of the hypothesis of Theorem 5.1.5 as the compactification does not appear in the dimension of the moduli space of Ricci-flat structures in Theorem 5.1.6. An advantage of our approach, however, is that we expect that it should generalize more easily to asymptotically conical manifolds.

5.11 Asymptotically conical manifolds

In this section, we discuss asymptotically conical manifolds. ALE manifolds are a special class of asymptotically conical (AC) manifolds. One could therefore ask if the proofs of

Theorem 5.1.5 and Theorem 5.1.6 extend to AC manifolds. In this section we address this question. While not giving an affirmative answer in either direction, we provide comments on the role of the special cone structure $\mathbb{R}_+ \times S^{n-1}/G$ of the ALE manifold in the proofs of Theorem 5.1.5 and Theorem 5.1.6. AC manifolds admit a more general cone structure $\mathbb{R}_+ \times L$, for a compact connected manifold L, then ALE manifolds. The cone structure is therefore what sets ALE manifolds apart from other AC manifolds. For an introduction to AC manifolds see for instance [BOY-GAL08, Chapter 9], [CON-HEI13] and [CON-HEI15]. We start by introducing AC manifolds and continue with a discussion of the possibility of extending various components of Chapter 5 to the AC setting.

Definition 5.11.1 (Tangent cone)**.** *Let (L, g) be a compact connected Riemannian manifold. A* tangent cone (C, h_0) *with* link L *is defined to be the manifold $C = \mathbb{R}_+ \times L$ with the product metric $h_0 = dr^2 \oplus r^2 g$ and where r is the distance from the apex in the metric completion of the cone.*

Definition 5.11.2 (Asymptotically conical manifold)**.** *Let (X, g) be a complete Riemannian manifold and let (C, h_0) be a tangent cone. We say that (X, g) is* asymptotically conical (AC) *with tangent cone (C, h_0) if there exists a diffeomorphism $\pi : X\backslash K' \to C\backslash K$ with K', K compact, such that $\nabla^k(\pi_* g - h_0) = O(r^{-\tau-k})$ for some $\tau > 0$ and all $k \geq 0$.*

The flat metric h_0 on $\mathbb{R}^n\backslash\{0\} \simeq \mathbb{R}_+ \times S^{n-1}$ can be written as $h_0 = dr^2 \oplus r^2 g_0$, for metric g_0 on S^{n-1} and distance function r from 0 in $\mathbb{R}^n$ with respect to h_0. An ALE manifolds asymptotic to $\mathbb{R}^n/G$ is therefore a special case of an AC manifolds with tangent cone $\mathbb{R}^n\backslash\{0\}/G$ and link S^{n-1}/G.

Remark 5.11.3 (Tools from analysis)**.** *Weighted Sobolev and Hölder spaces can be introduced for a general tangent cone $\mathbb{R}_+ \times S$ as was done for ALE manifolds in Section 5.2 (see [CON-HEI13, Section 2.2]). The basic set-up for the analysis on an AC manifold is therefore the same as it was on a ALE manifold. Theorem 5.2.8 is actually borrowed from an identical result about AC manifolds. Replacing Theorem 5.2.7 with [CON-HEI13, Theorem 2.11] will suffice for the application in Section 5.8. Note that the set of exceptional weights may change as it depends on the set of eigenvalues of the Laplacian on the link of the cone (see [CON-HEI13, Theorem 2.9]). This difference will affect the expression for the dimension of the moduli space of AC Ricci-flat structures as well.*

Remark 5.11.4 (AC differential geometry)**.** *ALE manifolds are crepant resolutions of* $\mathbb{C}^m/G$*, while AC manifolds are in general not. This means that we cannot apply [JOY00, Theorem 8.4.3] to conclude that* $H^{2,0}(X) = H^{0,2}(X) = 0$*. General AC manifolds may therefore admit skew-symmetric infinitesimal complex deformations.*

Remark 5.11.5 (Slice construction)**.** *We expect that the slice equation* $(2\delta_g + d\,tr_g)h = 0$ *would also be a good choice on AC manifolds. Whether all details of the slice construction generalize to the AC context remains to be checked.*

Remark 5.11.6 (Ricci-flat deformations)**.** *The defining equations of infinitesimal Ricci-flat deformations do not make use of the special ALE cone structure, so the space of infinitesimal ALE Ricci-flat deformations could be defined in the same way as in Definition 5.5.1 on AC manifolds. The key component of Theorem 5.5.3 is Theorem 2.1.11. The argument why we can apply it is based on the Fredholm properties of the involved operators, and carries over to AC manifolds as explained in Remark 5.11.3 with the appropriate changes. The same is true for the analysis involved in proving Lemma 5.5.2. The equations used for the splitting of the space of infinitesimal Ricci-flat deformations in Proposition 5.5.5 are not affected by the more general cone structure, so the proposition should hold if we replace the ALE manifold with an AC manifold. The identifications of the Hermitian and skew-Hermitian infinitesimal Ricci-flat deformations with Harmonic real* $(1,1)$*-forms and symmetric infinitesimal complex deformations respectively might require some more careful considerations in the AC context, especially for the skew-Hermitian infinitesimal Ricci-flat deformations.*

Remark 5.11.7 (Deformations of the pair $(\bar{X}, D)$)**.** *Section 5.6 about deformations of the pair* $(\bar{X}, D)$ *makes no reference to the special ALE cone structure and is readily applicable for AC manifolds that arise as the complement of a smooth divisor inside a compact manifold.*

Remark 5.11.8 (Stability results)**.** *The stability results in Section 5.7 are based on the Kodaira Embedding theorem, general theory of Chern classes, deformation theory of pairs* $(\bar{X}, D)$ *and on [TIA-YAU90] and [TIA-YAU91]. All of these ingredients apply to AC manifolds just as well as to ALE manifolds. Lemma 5.7.2 and Proposition 5.7.3 generalize directly to AC manifolds. Proposition 5.7.4 would probably require some more work. Also, we do not know if the ALE version by Deruelle and Kröncke (Theorem 5.10.2) of Kodaira* *and Spencer's stability result for Kähler metrics for complex deformations [KOD-SPE60, Theorem 15] can be extended to AC manifolds.*

Remark 5.11.9. *In Section 5.8 and Section 5.9 we use the material from the previous sections to prove Theorem 5.1.5 and Theorem 5.1.6 respectively. We do not know if the isometry group would act as a finite group on the premoduli space of AC Ricci-flat metrics. We make no additional use of the ALE cone structure in these two sections apart from that. Assuming a suitable AC version of the material leading up to Section 5.8 and Section 5.9 had been achieved and that the isometry group acts as a finite group on the premoduli space of AC Ricci-flat metrics, then the content of Section 5.8 and Section 5.9 should generalize from ALE manifolds to AC manifolds.*

Chapter 6

Examples: ALE manifolds

In this chapter, we consider examples of ALE Ricci-flat Kähler manifolds. The first examples were produced by Eguchi and Hanson in [EGU-HAN78], who constructed explicit ALE hyperkähler manifolds of complex dimension 2. Gibbon and Hawking generalized this construction in [GIB-HAW78]. In [CAL79, p.285] Calabi gave an explicit construction of ALE Ricci-flat Kähler manifolds of arbitrary dimension, which in dimension 2 recovers the Eguchi-Hanson metrics. Hitchin introduced in [HIT79] a hyperkähler quotient construction of ALE Ricci-flat Kähler manifolds. Kronheimer used it in [KRO89-1] to produce many more examples and in [KRO89-2] he classified all ALE hyperkähler manifolds. We start with the following example from [EGU-HAN78].

Example 6.0.10. *Consider* $\mathbb{R}^4\backslash\{0\} \simeq \mathbb{R}_+ \times S^3$ *with spherical coordinates* (r, x, y, z) *and metric*

$$g = f_1^2(r)dr^2 + r^2 f_2^2(r)((dx)^2 + (dy)^2) + r^2(dz)^2, \tag{6.1}$$

with coefficients

$$f_1(r) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1+(\frac{a}{r})^4}}\right) \quad and \quad f_2(r) = \sqrt{\frac{1}{2}\left(1+\sqrt{1-\left(\frac{a}{r}\right)^4}\right)},$$

where a *is some positive constant. Because of spherical symmetry in (6.1) we can quotient out by the group* $G = \{\gamma_1, \gamma_2\}$ *where* $\gamma_1 = id$ *and* γ_2 *is the identification of antipodal points* $(r, x, y, z) \mapsto (r, -x, -y, -z)$*. The function* f_1 *and* f_2 *tends to* 1 *for* $r \to \infty$ *so*

$(\mathbb{R}^4\backslash\{0\}/G, g)$ is asymptotic to $\mathbb{R}^4\backslash\{0\}/G = \mathbb{R}_+ \times S^3/G$ with the ALE metric $dr^2 + r^2\tilde{g}$ where $\tilde{g} = dx^2 + dy^2 + dz^2$ is metric on the link S^3. In [JOY00, Example 7.2.2] Joyce views this example from a complex perspective. Equip $\mathbb{C}^2$ with the complex coordinates (z_1, z_2). Let $G = \{id, \gamma\}$ where γ is the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$. Let (X, π) be the blow-up of $\mathbb{C}^2/G$ at 0. X is then a crepant resolution of $\mathbb{C}^2/G$. Define $f : X\backslash\pi^{-1}(0) \to \mathbb{R}$ by

$$f = \sqrt{r^4+1} + 2\log(r) - \log(\sqrt{r^4+1}+1).$$

Here r is the radius function $r(z_1, z_2) = \sqrt{|z_1|^2 + |z_2|^2}$ on X. Let $\omega = i\partial\bar{\partial}f$. It is a closed real $(1,1)$-form on $X\backslash\pi^{-1}(0)$ and it extends uniquely across the exceptional divisor. ω is the Eguchi-Hanson metric on X and it is Ricci-flat Kähler. Furthermore, for large r the function f satisfies $f(z_1, z_2) = r^2 + O(r^{-2})$, so $\omega = i\partial\bar{\partial}(r^2) + O(r^{-4})$. The standard Kähler metric h_0 on $\mathbb{C}^2$ has Kähler form $i\partial\bar{\partial}(r^2)$, so the Kähler metric g of ω satisfies $\pi_* g = h_0 + O(r^{-4})$. The metric g is therefore an ALE$_4$ Ricci-flat Kähler metric.

In [CON-HEI13, Section 4.1] Conlon and Hein explain the following example of an ALE manifold due to Calabi([CAL79]).

Example 6.0.11. *Consider the complex projective space $\mathbb{CP}^m$. Let L be the total space of the $U(1)$-bundle in the tautological bundle $\mathcal{O}(-1) \to \mathbb{CP}^m$ and let $L_k = L/\mathbb{Z}_k$. Denote by $C(L_m)$ the Kähler cone of L_m. As $L \simeq S^{2m-1}$ the cone $C(L_m)$ is $\mathbb{C}^m/\mathbb{Z}_m$. Denote the metric completion $C(L_m) \cup \{0\}$ by V_m. The total space X of the canonical bundle $K_{\mathbb{CP}^m}$ is Ricci-flat Kähler with a global holomorphic volume form and it is a crepant resolution of the Calabi-Yau cone V_m with link $S^{2m-1}/\mathbb{Z}_m$. It is therefore an ALE$_{2m}$ manifold. Also for this example has Joyce given an enlightening presentation ([JOY00, Example 8.2.5]). The m'th root of unity $\eta = e^{\frac{2\pi i}{m}}$ acts on $\mathbb{C}^m = (z_1, \ldots, z_m)$ by $\eta : (z_1, \ldots, z_m) \mapsto (\eta z_1, \ldots, \eta z_m)$. The group $G =< \eta >$ is a subgroup of $SU(m)$ and it acts freely on $\mathbb{C}^m/G\backslash\{0\}$. Let (X, π) be the blow-up of $\mathbb{C}^m/G$ and define $f : \mathbb{C}^m/G\backslash\{0\} \to \mathbb{R}$ by $f(x) = \sqrt[m]{r^{2m}+1} + \frac{1}{m}\sum_{j=0}^{m-1} \eta^j \log(\sqrt[m]{r^{2m}+1} - \eta^j)$, where $\log(\sqrt[m]{r^{2m}+1} - \eta^j)$ is defined by slicing $\mathbb{C}$ along the negative real axis and setting $\log(Re^{i\theta}) = \log(R) + i\theta$ for $R > 0$ and $\theta \in (-\pi, \pi)$. Define a $(1,1)$-form ω on $X\backslash\pi^{-1}(0)$ by $\omega = dd^c\pi^*(f)$. The form ω extends to a smooth closed positive $(1,1)$-form on all of X. Let g be the Kähler metric of ω. The metric g is in fact complete and Ricci-flat and has holonomy $hol(g) = SU(m)$.*

On $\mathbb{C}^m/G\backslash\{0\}$ *we can for large* r *rewrite* f *as*

$$f = r^2 - \frac{1}{m(m-1)} r^{2-2m} + O(r^{-2m}).$$

Note that the Kähler form of the Euclidean metric on $\mathbb{C}^m/G$ *is* $\omega_0 = dd^c(r^2)$*. The push-forward of* ω *is* $\pi_*\omega = \omega_0 - \frac{1}{m(m-1)} dd^c(r^{-2m+2}) + dd^c\eta$ *on* $\mathbb{C}^m/G\backslash\{0\}$ *where* $\eta = f - r^2 + \frac{1}{m(m-1)} r^{-2m+2}$*. For large* r *we further have* $\nabla^k\eta = O(r^{-2m-k})$ *on* $\mathbb{C}^m/G\backslash\{0\}$*. The metric* g *is therefore an* ALE_{2m} *Ricci-flat Kähler metric. In [JOY00, p.178] Joyce Conjectures that for* $m \geq 3$ *this is the only example of an ALE metric with holonomy* $SU(m)$ *that can be written down explicitly in coordinates.*

In the next example we consider a pair $(\bar{X}, D)$ which satisfies the hypothesis of Theorem 5.1.5 but does not admit any ALE Ricci-flat deformations.

Example 6.0.12. *Consider the pair* $(\mathbb{CP}^m, \mathbb{CP}^{m-1})$*. The hyperplane divisor* $\mathbb{CP}^{m-1}$ *is a smooth ample divisor. The line bundles* $K_{\mathbb{CP}^m}$ *and* $L_{\mathbb{CP}^{m-1}}$ *are related via* $K_{\mathbb{CP}^m} = \mathcal{O}(-m-1) = -(m+1)\mathcal{O}(1) = -(m+1)L_{\mathbb{CP}^{m-1}}$*. We see that*

$$\mathcal{T}_{\mathbb{CP}^m} \simeq \Omega^{m-1}_{\mathbb{CP}^m} \otimes \det(\mathcal{T}_{\mathbb{CP}^m}) \simeq \Omega^{m-1}_{\mathbb{CP}^m} \otimes \mathcal{O}_{\mathbb{CP}^m}(m+1).$$
$$\mathcal{T}_{\mathbb{CP}^m}(-\mathbb{CP}^{m-1}) \simeq \mathcal{T}_{\mathbb{CP}^m} \otimes -L_{\mathbb{CP}^{m-1}} \simeq \mathcal{T}_{\mathbb{CP}^m} \otimes \mathcal{O}_{C\mathbb{P}^m}(-1) \simeq \Omega^{m-1}_{\mathbb{CP}^m} \otimes \mathcal{O}_{\mathbb{CP}^m}(m).$$
$$\mathcal{T}_{\mathbb{CP}^{m-1}} \simeq \Omega^{m-2}_{\mathbb{CP}^{m-1}} \otimes \det(\mathcal{T}_{\mathbb{CP}^{m-1}}) \simeq \Omega^{m-2}_{\mathbb{CP}^{m-1}} \otimes \mathcal{O}_{\mathbb{CP}^{m-1}}(m).$$

By Kodaira vanishing theorem,

$$H^2(\mathbb{CP}^m, \mathcal{T}_{\mathbb{CP}^m}(-\mathbb{CP}^{m-1})) = H^2(\mathbb{CP}^m, \Omega^{m-1}_{\mathbb{CP}^m} \otimes \mathcal{O}_{\mathbb{CP}^m}(m)) = 0$$
$$H^2(\mathbb{CP}^{m-1}, \mathcal{T}_{\mathbb{CP}^{m-1}}) = H^2(\mathbb{CP}^{m-1}, \Omega^{m-2}_{\mathbb{CP}^{m-1}} \otimes \mathcal{O}_{\mathbb{CP}^{m-1}}(m)) = 0$$

as $2+(m-1) > m = \dim \mathbb{CP}^m$ *in the first case and* $2+(m-2) > m-1 = \dim \mathbb{CP}^{m-1}$ *in the second case. From [KAW78, Proposition 1] we have the following short exact sequence*

$$0 \longrightarrow \mathcal{T}_{\bar{X}}(-D) \longrightarrow \mathcal{T}_{\bar{X}}(\log D) \longrightarrow \mathcal{T}_D \longrightarrow 0.$$

The second level of the corresponding long exact sequence in sheaf cohomology reduces *to*

$$0 \longrightarrow H^2(\mathbb{CP}^m, \mathcal{T}_{\mathbb{CP}^m}(\log \mathbb{CP}^{m-1})) \longrightarrow 0,$$

Hence $H^2(\mathbb{CP}^m, \mathcal{T}_{\mathbb{CP}^m}(\log \mathbb{CP}^{m-1})) = 0$. *All infinitesimal complex deformations of the pair* $(\mathbb{CP}^m, \mathbb{CP}^{m-1})$ *are therefore integrable by [KAW78, Corollary 4]. The complement of the divisor* $\mathbb{CP}^{m-1}$ *is* $\mathbb{C}^m$ *and it trivially admits an ALE*$_{2m}$ *Ricci-flat Kähler metric. The space* $\mathbb{C}^m$ *is naturally a crepant resolution of* $\mathbb{C}^m$*, so* $H^{0,2}(\mathbb{C}^m) = 0$ *by [JOY00, Theorem 8.4.3]. The dimension of the premoduli space of ALE*$_{2m-1}$ *Ricci-flat Kähler metrics is now given by*

$$\dim \mathcal{H}^{1,1}_{\mathbb{R}}(\mathbb{C}^m, J) - 1 + 2\dim_{\mathbb{C}} H^1(\mathbb{C}^m, \mathcal{T}_{\mathbb{C}^m}).$$

On Euclidean space the Betti numbers b^i *vanish for all* $i \geq 1$*, so* $\mathcal{H}^{1,1}_{\mathbb{R}}(\mathbb{C}^m, J) = 0$. *There are therefore no Hermitian infinitesimal Ricci-flat deformations. Cartan's theorem B says that for a Stein manifold* X *and a coherent sheaf* $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ *for all* $i \geq 0$. *The manifold* $\mathbb{C}^m$ *is Stein and the sheaf of sections of a holomorphic vector bundle is locally free, hence coherent, so by Cartan's theorem B,* $H^1(\mathbb{C}^m, \mathcal{T}_{\mathbb{C}^m}) = 0$. *All cohomology groups involved in the expression for the dimension of the space of infinitesimal ALE*$_{2m-1}$ *Ricci-flat deformations therefore vanish. The standard Euclidean metric on* $\mathbb{C}^m$ *therefore does not admit any ALE*$_{2m-1}$ *Ricci-flat deformations.*

We have been unable to find more interesting applications of Theorem 5.1.5 and Theorem 5.1.6, but as we have discussed in Section 5.11, Theorem 5.1.5 and Theorem 5.1.6 can also be seen as a step on the way towards a version of Koiso's results for asymptotically conical manifolds.

Concluding remarks

An ideal candidate for a distinguished metric should preferably always exist, be unique and have interesting properties. While Koiso's results and the Calabi conjecture do not provide actual existence and uniqueness of Einstein metrics on general compact manifolds, they do tell us that there are in some sense not too many Einstein structures on a compact Kähler manifold and that Einstein metrics always exist on compact Kähler manifold with vanishing first Chern class. This is part of the justification for promoting Einstein metrics as candidates for the role of distinguished metrics on compact manifolds given in [BES87]. In this thesis we have studied two ways of generalizing results by Koiso from [KOI83].

In Chapter 3 we studied the effect of introducing quotient singularities on Koiso's results. It turned out that this effect was minor and we managed to prove a satisfying generalization. For a compact Ricci-flat Kähler orbifold we showed that orbifold Ricci-flat deformations of orbifold Ricci-flat Kähler metrics are Kähler possibly with respect to a perturbed complex structure. We also found an expression for the dimension of the moduli space of orbifold Ricci-flat structures in a neighbourhood of an orbifold Ricci-flat Kähler structure. In addition to this, we showed that the proof of the Calabi conjecture goes through on compact orbifolds.

Another way to generalize Koiso's results is to relax the compactness assumption to allow complete metrics on non-compact manifolds. In Chapter 5 we studied a particular class of complete non-compact manifolds known as asymptotically locally Euclidean manifolds (ALE). For an ALE Ricci-flat Kähler manifold X that arises as $X = \bar{X} \setminus D$ for a compact Kähler manifold $\bar{X}$ and a smooth ample divisor D satisfying $K_{\bar{X}} = -\beta L_D$ for some $\beta \geq 1$ we showed that if all infinitesimal deformations of the pair $(\bar{X}, D)$ are integrable, then ALE Ricci-flat deformations of an ALE Ricci-flat Kähler metric is Kähler possibly with respect to a perturbed complex structure. We also found an expression for the moduli space of ALE Ricci-flat deformations in a neighbourhood of a ALE Ricci-flat

Kähler structure.

Based on the work we did in Chapter 3 and Chapter 5 it seems reasonable to view Einstein metrics as a potential candidate for the role of a distinguished metric also on orbifolds and ALE manifolds.

ALE manifolds are a special class of asymptotically conical manifolds. Conlon and Hein proved in [CON-HEI15] an asymptotically conical version of the Calabi conjecture. In Section 5.11 we discussed the possibility of extending Theorem 5.1.5 and Theorem 5.1.6 from ALE manifolds to asymptotically conical manifolds. It would be interesting to pursue such a generalization.

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Notation Index