DOI: 10.1017/psa.2022.10

This is a manuscript accepted for publication in *Philosophy of Science*.

This version may be subject to change during the production process.

On internal structure, categorical structure, and representation*

Neil Dewar†

Abstract

If categorical equivalence is a good criterion of theoretical equivalence, then it would seem that if some class of mathematical structures is represented as a category, then any other class of structures categorically equivalent to it will have the same representational capacities. [Hudetz, 2019a] has presented an apparent counterexample to this claim; in this note, I argue that the counterexample fails.

*I am very grateful to Lia den Daas and John Dougherty for comments on and discussion of an earlier draft of this paper; to two anonymous referees for their detailed remarks; and to Laurenz Hudetz, Thomas Barrett, and Hans Halvorson for many conversations about the issues discussed here. Gefördert durch die Deutsche Forschungsgemeinschaft – Projektnummer 392413352.

†Faculty of Philosophy, University of Cambridge; email: nad42@cam.ac.uk

1 Introduction

A central topic in the philosophy of science is that of *theoretical equivalence*: the relationship that holds between two theories when, roughly, they 'say the same things about the world'. Recent work in formal philosophy of science has suggested that an illuminating way of getting a handle on this can be to present theories as categories of models, and investigate whether those categories are equivalent (in the sense of admitting a full, faithful, and essentially surjective functor between them). However, the philosophical foundations of such work remain somewhat opaque. In particular, it is not clear whether a category of models is really capable of capturing enough of a theory's structure to make categorical equivalence an interesting or useful notion by itself—even insofar as we are only considering formal criteria of equivalence between theories.

A specific version of this concern has recently been articulated by Hudetz (2019a). He notes that categorical equivalence between categories of models depends only on how models are related to one another by the morphisms of the category. He argues that this means categorical equivalence will (in general) fail to take into account the internal structure of those models—even where that internal structure plays a representationally significant role. He then observes that the category $\mathsf{FinVect}$ of finite-dimensional vector spaces with linear mappings as morphisms is categorically equivalent to the category Mat of natural numbers with real-valued matrices as morphisms. But, he argues,

> Whenever vector spaces are used to model features of real-world systems, we cannot simply replace them with natural numbers. Vector spaces possess internal structure: a set of vectors, the operations of vector addition and scalar multiplication, a zero vector (and an underlying field of scalars). In

> applications, some of these features serve representational purposes. For example, vectors might be used to represent forces acting on objects, and the net force on an object may be represented by a sum of vectors. It is utterly impossible that a natural number n could serve the same representational purposes as a vector space of dimension n. The number n simply lacks the internal structure necessary for fulfilling such representational roles. (Hudetz 2019a, 52)

The purpose of this note is to argue that this example fails in its intended purpose. First, I sketch how the internal structure of a vector space can be recovered from the structure of the category of (finite-dimensional) vector spaces; and hence, how it can be recovered from the structure of any categorically equivalent category, such as the category of natural numbers with matrices as morphisms. I then consider two ways to support the conclusion—strange as it may sound—that a natural number and a vector space do, in some appropriate sense, have the same representational capacities: one which depends on this observation about recovery, and one which does not.

2 From categories to vectors

It is well-known that one can recover the internal structure of a vector space from the category $\mathsf{FinVect}$.[1] What I mean by this is the following. Consider the normal definition of a vector space, in (broadly) model-theoretic terms, as a structure in signature $\{\cdot, +\}$

1. Indeed, I learned of this result from Hudetz (2019b). For further discussion of this result, see (Spivak 2017); I thank an anonymous reviewer for drawing my attention to this paper. John Dougherty has also observed (personal communication) that the result here is a special case of the Mitchell-Freyd embedding theorem.

obeying the vector-space axioms. Consider next the category FinVect as a two-sorted structure of signature $\{\circ\}$. It is possible to stipulate definitions of the symbols $\cdot$ and $+$ in terms of $\circ$, in such a way that for any vector space V, V is isomorphic to some substructure of the definitional expansion of FinVect (that is, that there is an injection preserving $\cdot$ and $+$ from V to FinVect).

In brief, how this works is as follows. Given two objects of FinVect (i.e. vector spaces) U and V, their direct sum $U \oplus V$ is a *biproduct*: it is both a product and a coproduct.[2] Since FinVect has all finite biproducts, it is possible to define a binary operation known as "addition" on the arrows between any two objects, purely in terms of the categorical structure. When we do so, we find that this operation does indeed coincide with usual pointwise definition of addition of linear maps. Hence, every linear map possesses an additive inverse. This means that the category FinVect is what is known as an *additive category*.

By using this definition of addition, defining multiplication as precomposition, and defining order in terms of addition and multiplication, we can equip the set of arrows from the one-dimensional vector space $\mathbb{1}$ to itself with the structure of an ordered field; hence, we are able to identify that set as canonically isomorphic to $\mathbb{R}$. We then define scalar multiplication on the set of arrows from $\mathbb{1}$ to any vector space V as precomposition (with some arrow from $\mathbb{1}$ to itself); together with the above definition of addition, it can then be shows that this set has the structure of a vector space, and indeed is isomorphic to V. Hence, we are able to recover the "internal structure" of the objects in FinVect from its categorical structure.

Mat has the same categorical structure as FinVect (since it is categorically equivalent

2. For a proof, see e.g. (Geroch 1985, chap. 11).

to it), and so we can use the same reasoning to recover vectors within the category Mat: in particular, since being an additive category is a property that is statable in purely category-theoretic terms, and since category-theoretic properties are preserved across categorical equivalence, we know that Mat is an additive category. More concretely, the biproduct of m and n in Mat is $m + n$, and the categorical notion of addition (of arrows in the category) turns out to be standard matrix addition.

Using this, we show that the set of 1×1 matrices is canonically isomorphic to $\mathbb{R}$, and that the set of $n \times 1$ matrices constitutes an n-dimensional vector space. Of course, we knew this already: it is hardly a surprise to learn that column matrices can represent vectors! The point, though, is that although we know how matrix algebra can be used to provide an instantiation of linear algebra, it is less obvious that matrix algebra can, in turn, be defined up out of purely categorical notions. That demonstrates the sense in which the categorical structure here is richer than one might have expected.

3 Discussion

I turn now to Hudetz's claim that 'it is utterly impossible that a natural number n could serve the same representational purposes as a vector space of dimension n'. Let us suppose—as Hudetz suggests—that we wish to model the forces on an object by using a three-dimensional vector space V, with vector summation corresponding to superposition of forces. Can we indeed 'replace' the vector space V by the natural number 3? Per the above, this replacement would presumably mean using 3×1 matrices to represent the various forces, with superposition of forces represented by matrix addition; does this properly 'count' as using the natural number 3 to represent the empirical situation?

Fully answering this question would mean engaging with the extensive literature on scientific representation, which is not going to be possible in a short discussion note like this.[3] To make things manageable, then, let us restrict our attention to so-called 'structuralist' accounts of representation: that is, accounts that 'take models to be structures, which represent their target systems in virtue of there being some kind of *morphism* (isomorphism, partial isomorphism, homomorphism, ...) between the two.'[4] The structuralist account has its critics; but it is a reasonably popular approach, and certainly seems like a natural place to try and get a grip on the significance of internal structure for the purposes of representation. For the sake of simplicity (and ease of connection with the literature), I will focus on the case where the relevant morphism is taken to be an isomorphism.

Evidently, the content of the structuralist account is then going to depend upon how the notions of structure and isomorphism are understood. Frigg and Nguyen (2017) observe that these are usually explicated in a sense familiar from set theory and mathematical logic, through definitions like the following:

> A structure S in that sense [...] is a composite entity consisting of the following: a non-empty set U of objects called the domain (or universe) of the structure and a non-empty indexed set R of relations on U. (66)
>
> Two structures $S_a = \langle U_a, R_a \rangle$ and $S_b = \langle U_b, R_b \rangle$ are *isomorphic* iff there is a mapping $f : U_a \to U_b$ such that (i) f is one-to-one (bijective) and (ii) f preserves the system of relations in the following sense: the members $a_1, \ldots, a_n$ of U_a satisfy the relation r_a of R iff the corresponding members

3. For an overview of that literature, see (Frigg and Nguyen 2018).
4. (Frigg and Nguyen 2018)

$b_1 = f(a_1), \ldots, b_n = f(a_n)$ satisfy the relation r_b of R_b, where r_b is the relation corresponding to r_a. (Frigg and Nguyen 2017, 68)

How the isomorphism relation is set up between the model and the target system is then a further point of controversy, with several proposed answers; perhaps the simplest suggestion is that target systems instantiate structures, by virtue of having parts that stand in appropriate relations to one another, and that the isomorphism (if it exists) holds between the model and this physically instantiated or realised structure.[5] In the context of our example, the 'parts' of the target system are presumably the individual forces: these stand in structure-realising relations (e.g. the ternary relation "x is the superposition of y and z"), and so form a structure to which the model is—hopefully—isomorphic.

I conjecture that this account of representation, or something like it, is at work in Hudetz's assertion that 3 cannot serve the same representational purposes as the three-dimensional vector space V: the problem, so the thought goes, is that 3 is not capable of standing in the appropriate kind of isomorphism relation to the physical situation. It seems to me that there are two ways in which this assertion could be backed up. On the one hand, it might be maintained that 3 simply is not a 'structure' in the sense laid out above (i.e. a composite entity consisting of a set of objects equipped with relations): it is an element of a certain structure or structures (e.g. the natural or real numbers), but is not itself a structure. Alternatively, one could admit that 3 is or may be taken to be a structure, but argue nevertheless that it is not a structure which is isomorphic to V. For instance, perhaps we can take 3 to have the structure of the

5. See (Frigg and Nguyen 2017, §5.4) for more details, and other ways in which the isomorphism relation could be understood.

relevant von Neumann ordinal $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ or the Zermelo ordinal $\{\{\{\varnothing\}\}\}$; but these are both finite sets, and so no bijection could be set up between either of them and the infinite-cardinality structure V.

Thus, in resisting Hudetz's assertion, we must also resist this account of representation.[6] But if one is attracted to category-theoretic accounts of the structure of scientific theories, then this is a very natural move: the above is explicitly based upon resources from set theory, after all.[7] From a more categorical perspective, it is just as true that the three-dimensional vector space V, as an object of FinVec, is not a composite entity consisting of a set of objects equipped with relations. Thus, what makes the object V an appropriate representation of the physical situation is not the existence of a bijection between the parts of that situation and the elements of V. What does make it an appropriate representation is a somewhat tricky question: here are two possible answers.

First, the 'conservative' answer. Although V does not possess elements in the set-theoretic sense, it does possess elements in a generalised sense—if 'elements' are understood as morphisms from some suitable 'test object' (as is often done in category theory), then we can take the elements of V to be the morphisms from $\mathbb{1}$ to V that were employed above. Moreover, we have seen that the vector-space structure of V is recoverable from the categorical facts about these morphisms. It is therefore coherent to require (in order that V be a representation) that there be an appropriate isomorphism

6. Thanks to an anonymous referee for making the need for a different account of representation clear to me.

7. Landry (2007) also pushes back against the supremacy of set-theoretic ways of understanding 'structure' and '(iso)morphism'—although Landry would, I think, be similarly wary of granting category theory (rather than set theory) a monopoly on how those terms should be understood. In a related vein, see (Barrett 2020) for a critique of isomorphism as the right way to explicate the notion of 'having the same structure'.

between the physically instantiated structure and this vector-space structure. So we can borrow the structuralist's account of what makes for representation, modulo the appeal to a more generalised understanding of what the 'elements' of a structure are.

And, of course, this more generalised understanding of elements opens the door to admitting the number 3 (at least qua element of Mat) as an appropriate representation of the physical situation. For, it admits of elements in just the same manner that V does; and as we have seen above, those elements (namely, 3×1 matrices) are just as good at being correlated to forces for representational purposes as the elements of V are. Of course, qua an object of a different category, 3 will not carry this representational capacity. For instance, 3 will not be suitable for representing the physical situation as an object of an ω-sequence Nat (which, being a partial order, we may regard as a category). But from the present perspective this is a feature, not a bug. Regarded as an object of Nat, the number 3 is also not equivalent to a three-dimensional vector space, since there is no categorical equivalence between Nat and FinVec. For that matter, there is no categorical equivalence between Nat and Mat: so the number 3 is not always equivalent to itself, nor does it always have the same representational capacities as itself—anymore than the sign 'die' has the same representational capacities as a word of English and as a word of German.

This account invites the following objection: that in such a case, it is not 3 itself that is doing the representing, but rather some substructure of the category Mat (since the forces are being correlated to certain elements of Mat, i.e. the morphisms from 1 to 3).[8] However, this objection only works if we insist that elements must be understood in the set-theoretic, rather than the category-theoretic, fashion—and to make such an

8. I thank an anonymous reviewer for pressing this objection.

insistence is to beg the point at issue. On category-theoretic accounts of scientific theories, we should expect the representational capacities of any object to be relative to the category which it is presented as being a part of; just as on the set-theoretic account, the representational capacities of a certain set are relative to what structures and operations it has been equipped with.

Let us turn now to the second, more radical, answer to the question of what it is that could make an object of a category (such as V or 3) an appropriate representation of a certain concrete physical situation. This will be somewhat more vague and sketchy; but I find it intriguing, so I include it in case other readers of this journal find it stimulating to consider. This more radical answer gives up on the idea that the representation relation is based upon any kind of correspondence between the parts, or pieces, or components of the situation and the representation. Rather (on this view) the situation is appropriately represented by V because it is a situation *of a certain kind*; and structural features of the whole collection of situations of that kind make them apt to be represented by the category of which V is an object. For instance, if we can consider all the situations featuring forces on an object as a category, then we can propose that these situations can be represented by vector spaces just in case there is an appropriate relationship between that category and the category FinVec. An isomorphism of categories is not a very natural candidate: it would be more natural to require that the categories be equivalent, or perhaps merely that one category be equivalent to a subcategory of the other. Evidently, if we do so, then representational appropriateness will be preserved across categorical equivalence: that is, $\mathsf{FinVect}$ will be appropriate for representation if and only the equivalent category Mat is. Note that this holds independently of the considerations in §2: in other words, the reconstruction of vectors in the categories FinVec or Mat is only only playing a dialectical role in the first (more conservative) answer.

This second proposal foregrounds the significance of *modal* structure. In effect, the idea is to consider a category of possible situations (rather than an individual situation), and hold that representation arises when that category of possibilities is appropriately similar to the category being used for representation.[9] We might, therefore, think of it as a special case of a more general account of how state- or possibility-spaces represent their targets. In such a space, each individual possibility is represented by something lacking internal structure: a point of a configuration space, a vector in a Hilbert space, an element of a Boolean algebra, etc. However, that does not make such a space representationally hopeless, since the space itself can carry a rich enough structure to support our representational practice.[10] This structure must, however, be understood as imputing properties or relations to the individual possibilities (as wholes), rather than to whatever individuals might populate one possibility or another.[11]

This then raises the question of what (if anything) might be significant about the case where that modal space is represented as a category. Here we should remember that, by virtue of the Yoneda Lemma, the objects of a category are fully determined by their morphisms: as Spivak puts it, 'categories can only model "relationally-determined" subjects, subjects in which each object of study is ontologically determined by its

9. An example of such a category might be the category of models of some theory: the objects of the category represent the different possibilities according to that theory (i.e. the "nomological possibilities", if we take the theory to be expressing a certain set of natural laws).

10. This point is made forcefully by, for instance, Wallace and Timpson (2010) and Wallace (2021) in the context of the debate over how to interpret the quantum state. See also (van Fraassen 1987) for discussion of the representational significance of state-spaces.

11. Or at least, it does not impute such properties to individuals directly. This is not to say that we might not be able to recover properties of individuals, by reading them off from the properties of possibilities: for instance, from a point of N-fold configuration space we can recover the locational properties of the N particles that, presumably, populate the possibility which that point represents.

relationships to the others.'[12] So the modal spaces that category theory is capable of describing will be those where the relationships *between* possibilities encapsulate everything there is to know about those possibilities. This suggests a potentially very novel way of thinking about possibility and ontology: one where the differences between possibilities, rather than the intrinsic character of those possibilities, are primary.[13] Clearly, however, working out such a view will have to be a project for another time.

4 Conclusion

So what, then, of Hudetz's argument? Here is how it seems to me that things stand. The more conservative account of categorical representation, together with the argument of §2, leads to the conclusion that the example adduced by Hudetz does not do quite what he wants. However, that is not to say that other examples would not pose problems for that account. In particular, if one had a category for which the internal structure was *not* recoverable (i.e. for which there was no analogue of §2's argument), then the conservative account will struggle to push back against the conclusion Hudetz wishes to draw from this (that categories do not always capture everything that is representationally relevant). Indeed, this would appear to be so for some of the other examples discussed by Hudetz: for instance, the (trivial) categories for theories with unique, rigid models.

On the other hand, the radical account—being independent of the argument in §2—provides a more general way to resist Hudetz's conclusion. The price of this is that that account is still in need of much more development. So either way, the above hardly

12. (Spivak 2017, 393)

13. At least, novel in the analytic tradition: the 'differential ontology' of Deleuze (1994) advocates something along these lines.

constitutes a refutation of Hudetz's argument in the passage quoted above; still less does it invalidate the main point of Hudetz's paper (to show how considerations of definability and categorical structure may be fruitfully combined with one another). As we have seen, however, it does provide a route in to thinking about scientific representation in ways that are somewhat novel and unorthodox. For that reason, if nothing else, this little example provides food for thought.

References

Barrett, Thomas William. 2020. "Structure and Equivalence." *Philosophy of Science* 87 (5): 1184–1196.

Deleuze, Gilles. 1994. *Difference and Repetition.* Translated by Paul Patton. 350 Pages. New York: Columbia University Press, May. ISBN: 978-0-231-08158-0.

Frigg, Roman, and James Nguyen. 2017. "Models and Representation." In *Springer Handbook of Model-Based Science,* edited by Lorenzo Magnani and Tommaso Bertolotti. Http://philsci-archive.pitt.edu/id/eprint/12834. Berlin; New York: Springer.

———. 2018. "Scientific Representation." In *The Stanford Encyclopedia of Philosophy,* Winter 2018, edited by Edward N. Zalta. Https://plato.stanford.edu/archives/win2018/entries/scientific-representation/. Berlin; New York: Metaphysics Research Lab, Stanford University.

Geroch, Robert. 1985. *Mathematical Physics.* Winter 2018, edited by Edward N. Zalta. Chicago Lectures in Physics. Https://plato.stanford.edu/archives/win2018/entries/scientific-representation/. Chicago: University of Chicago Press.

Hudetz, Laurenz. 2019a. "Definable Categorical Equivalence." Edited by Edward N. Zalta. Https://www.journals.uchicago.edu/doi/abs/10.1086/701047, *Philosophy of Science* (Chicago), Chicago Lectures in Physics, 86, no. 1 (January): 47–75. ISSN: 0031-8248. https://doi.org/10.1086/701047.

Hudetz, Laurenz. 2019b. "Defining Internal Structure in Terms of Morphisms." In *Foundations of Categorical Philosophy of Science,* edited by Edward N. Zalta, 86:47–75. Chicago Lectures in Physics 1. Https://www.journals.uchicago.edu/doi/abs/10.1086/701047. LMU Munich: University of Chicago Press, January. https://doi.org/10.1086/701047.

Landry, Elaine. 2007b. "Shared Structure Need Not Be Shared Set-Structure." Edited by Edward N. Zalta. Http://link.springer.com/article/10.1007/s11229-006-9047-7, *Synthese* (LMU Munich), Chicago Lectures in Physics, 158, no. 1 (January): 1–17. ISSN: 0039-7857, 1573-0964. https://doi.org/10.1007/s11229-006-9047-7.

Spivak, David I. 2017b. "Categories as Mathematical Models." In *Categories for the Working Philosopher,* Winter 2018, edited by Elaine Landry, 158:381–401. Chicago Lectures in Physics 1. Http://link.springer.com/article/10.1007/s11229-006-9047-7. Oxford: Oxford University Press, January. https://doi.org/10.1007/s11229-006-9047-7.

van Fraassen, Bas C. 1987b. "The Semantic Approach to Scientific Theories." In *The Process of Science: Contemporary Philosophical Approaches to Understanding Scientific Practice,* Winter 2018, edited by Nancy Nersessian, 158:105–124. Chicago Lectures in Physics 1. Http://link.springer.com/article/10.1007/s11229-006-9047-7. Dordrecht: Martinus Nijhoff, February. ISBN: 978-90-247-3425-2. https://doi.org/10.1007/s11229-006-9047-7.

Wallace, David. 2021b. "Against Wavefunction Realism." In *Current Controversies in the Philosophy of Science,* Winter 2018, edited by Nancy Nersessian, 158:63–74. Chicago Lectures in Physics 1. Http://link.springer.com/article/10.1007/s11229-006-9047-7. New York: Routledge, February. ISBN: 978-90-247-3425-2. https://doi.org/10.1007/s11229-006-9047-7.

Wallace, David, and Christopher G. Timpson. 2010b. "Quantum Mechanics on Spacetime I: Spacetime State Realism." Edited by Nancy Nersessian. Http://bjps.oxfordjournals.org/content/61/4/697, *The British Journal for the Philosophy of Science* (New York), Chicago Lectures in Physics, 61, no. 4 (January): 697–727. ISSN: 0007-0882, 1464-3537. https://doi.org/10.1093/bjps/axq010.