Integrability from Chern-Simons Theories

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To Beryl and Varlon, in loving memory.

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

Roland Bittleston
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Above all, I am grateful to Holly for sharing this journey with me from start to finish. I look forward to what we will accomplish together next.

Abstract

This thesis details my work exploring connections between integrable systems and Chern-Simons theories. It is divided into two parts. The first concerns the application of 4d Chern-Simons theory to describe integrable models with boundary, while the second concerns relations between holomorphic Chern-Simons theory on twistor space, 4d Chern-Simons theory and the anti-self-dual Yang-Mills equations.

Part one opens with a review of 4d Chern-Simons theory, including a discussion of its connections to both quantum and classical integrable systems. It then turns to the results of this thesis concerning the application of 4d Chern-Simons theory to generate solutions of the boundary Yang-Baxter equation. They include: defining the boundary analogue of a quasi-classical R-matrix and classical r-matrix; realising K-matrices as the vacuum expectation values of Wilson lines in 4d Chern-Simons theory on a $\mathbb{Z}_2$ orbifold; deriving the order $\hbar$ contribution to a K-matrix in the rational case and verifying that it obeys the boundary Yang-Baxter equation to second order in $\hbar$; determining the OPE of bulk and boundary Wilson lines; demonstrating that boundary line operators are labelled by representations of twisted Yangians; giving the gauge theory realisation of boundary unitarity and the Sklyanin determinant; proving the uniqueness of the rational K-matrix; obtaining explicit formulae for the order $\hbar$ contributions to trigonometric and elliptic K-matrices and matching them to examples in the literature.

Part two begins with a review of twistor theory. This is followed by the results of this thesis concerning the connections between holomorphic Chern-Simons theory on twistor space, 4d Chern-Simons theory and the anti-self-dual Yang-Mills equations. They include: showing that holomorphic Chern-Simons theory on twistor space for a meromorphic measure descends to an integrable theory on 4d spacetime; extending these results to indefinite signatures; identifying 4d Chern-Simons theory as the quotient of a 6d Chern-Simons theory on twistor correspondence space by an appropriate lift of the 2d translation group on spacetime; quotienting holomorphic Chern-Simons theory on twistor space by a 1 dimensional group of translations to obtain a 5d Chern-Simons theory on minitwistor correspondence space describing the Bogomolny equations.

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Nomenclature

Acronyms / Abbreviations

ASDYM Anti-self-dual Yang-Mills

BYBE Boundary Yang-Baxter equation

CBYBE Classical boundary Yang-Baxter equation

CIFT Classical integrable field theory

CS_d d dimensional Chern-Simons

HCS Holomorphic Chern-Simons

CYBE Classical Yang-Baxter equation

PCM Principal chiral model with Wess-Zumino-Witten term

QIFT Quantum integrable field theory

UEA Universal enveloping algebra

VEV Vacuum expectation value

WZW_d d dimensional Wess-Zumino-Witten model

YBE Yang-Baxter equation

Chapter 1

Introduction, summary and background

1.1 Introduction

In his 1989 classic paper 'quantum field theory and the Jones polynomial' [166] Witten demonstrated that a particular quantum field theory in 3 dimensions known as Chern-Simons theory (CS_3) can be used to compute knot, or more accurately ribbon, invariants.

It is well known that certain manipulations of knots are closely related to scattering processes in 2d integrable models. This connection is most easily understood by considering the following diagram.

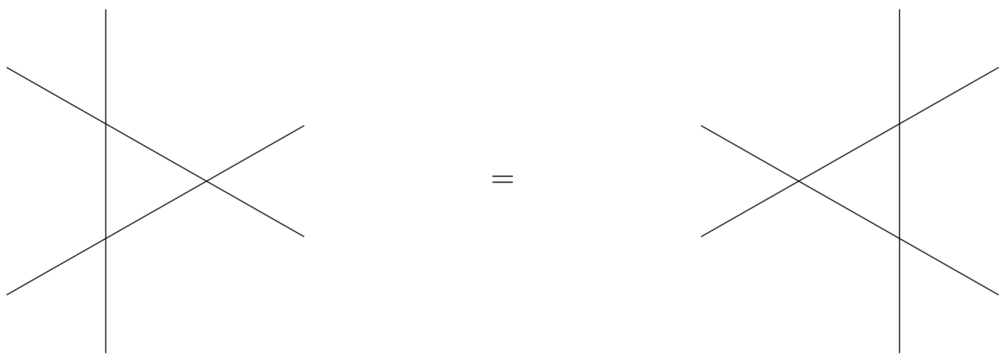


Fig. 1.1 Type III Reidemeister move or Yang-Baxter equation?

It can be interpreted in two different ways.

- We can view the three lines in the figure as being at different heights. In going from the left hand side to the right the vertical line has been pushed past the crossing point of the other two. This is known as a type III Reidemeister move in knot theory.

- We can interpret the lines as trajectories of particles in a 2d relativistic quantum field theory, and the crossings as scattering events. For this interpretation to make sense it is essential that two incoming excitations always interact to give two outgoing excitations. For a generic 2d relativistic field theory there is no reason for this to be the case. Remarkably, however, there exist many non-trivial theories with enhanced symmetries which guarantee this behaviour. These symmetries also ensure that the scattering of n particles is reducible to pairwise scattering events. The two sides of the above figure illustrate different decompositions of the 3-particle S-matrix into a combination of 2-particle S-matrices. The equality ensures that these two decompositions give the same result. This condition is known as the Yang-Baxter equation (YBE).

A 2d relativistic quantum field theory with enhanced symmetry which ensures that scattering is reducible to 2-particle scattering events is known as a quantum integrable field theory (QIFT).

The YBE plays a second role in the theory of integrable systems. There exist certain 1 dimensional quantum spin chains whose interactions are encoded in a solution to the YBE referred to as an R-matrix. The Hilbert space of states and Hamiltonian of the spin chain can be built up by concatenating R-matrices, and integrability is assured by the YBE. Broadly speaking, the spectrum of such spin chains can be determined using a method known as the algebraic Bethe ansatz.

Given the similarity between the type III Reidemeister move and the YBE, it is natural to anticipate the existence of a Chern-Simons type theory describing integrable models. Indeed, Witten himself was aware of this possibility and sought such a description [165]. It was not until 2013 that the correct theory was identified by Costello in his paper 'supersymmetric gauge theory and the Yangian' [42]. It now goes by the name 4d Chern-Simons (CS_4) theory. (The Lagrangian for CS_4 also appeared briefly in Nekrasov's PhD thesis [134].) In this paper Costello demonstrated that the partition function of the Heisenberg spin chain can be obtained as the vacuum expectation value (VEV) of a particular configuration of Wilson lines in CS_4. Strikingly, the YBE follows immediately from the symmetries of the theory. In a subsequent pair of papers the connections between CS_4, quantum integrable spin chains and the quantum groups which govern their solubility were further developed [50, 51]. Much of the content of these papers will be reviewed as background material for this thesis.

In the third paper in this series [53] it was argued that by inserting order and disorder defects into CS_4 and then eliminating the gauge field using its classical equations of motion one could obtain a plethora of 2d classical integrable field theories (CIFTs). These include numerous σ-models on groups, cosets and Kähler manifolds. Again, integrability is manifest from the 4d perspective: the dynamical field of CS_4 encodes the Lax of the corresponding 2d CIFT. This observation sparked a series of rapid developments in the field. Most notable among these was the demonstration in [158] that the Hamiltonian formulation of CS_4 leads to affine Gaudin models. Many more 2d CIFTs have also been brought under the umbrella of CS_4, including the λ- and η-deformations [60], integrable $\mathcal{E}$-models [102] and the Metsaev-Tseytlin σ-model [49]. There have also been tentative steps towards an understanding of 2d QIFTs using CS_4 in the form of Kondo problems and the related ODE/IM correspondence [79, 78].

In this thesis I outline my contributions to the field. It is split into two main parts: the first addresses how integrable models with boundary can be incorporated into CS_4, and the second connects recent developments in describing CIFTs using CS_4 to an older perspective which views them as symmetry reductions of the anti-self-dual Yang-Mills (ASDYM) equations. These are based on the papers [29, 30] and [31] respectively.

We begin by briefly summarising the content of the thesis in section 1.2, before covering essential background in section 1.3. All work was done in collaboration with my supervisor Dr David Skinner.

1.2 Summary of thesis

Before we begin, it is first necessary to review CS_4 and some of its connections to integrable systems. To this end, section 1.3 provides an extensive review of [50], referencing some results from [51] when needed. Towards the end of the section we also provide a brief introduction to [53].

In brief, we first present the YBE and review the definition of a quasi-classical R-matrix. Classical CS_4 is then introduced, which is followed by a longer subsection explaining how R-matrices can be generated as the VEVs of crossing Wilson lines. To determine the order $\hbar$ contribution to the R-matrix it is necessary to first fix the gauge and identify a suitable propagator. This is done in the simplest case, and the standard rational r-matrix is recovered. Using the uniqueness results of [51] the corresponding quasi-classical R-matrices in the fundamental representation of $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ are identified to all orders in $\hbar$.

The next subsection focuses on the emergence of the Yangian in CS_4. We review how the OPE of line operators determines the coproduct, and sketch the origin of the algebra deformation from a 2-loop diagram. The framing anomaly is then discussed, including the correction it necessitates to the configuration generating an R-matrix. Finally, the realisation of the RTT presentation of the Yangian in CS_4 is surveyed.

The extension to the trigonometric and elliptic cases is reviewed in the succeeding subsection. Here we slightly extend the results of [50, 51]. In the trigonometric case we argue that the standard Jimbo R-matrix in the fundamental of $\mathfrak{sl}_n(\mathbb{C})$ arises from CS_4 for the simplest choice of boundary conditions. We also note that modifying these has the effect of 'twisting' the R-matrix. In the elliptic case we determine the classical r-matrix associated to $\mathfrak{sl}_n(\mathbb{C})$ for all $n \geq 2$, and match it to Belavin's elliptic R-matrix.

We conclude the background section with a brief discussion of how CIFTs can be obtained from CS_4 in the presence of disorder defects. We focus on the case leading to the principal chiral model with Wess-Zumino-Witten term (PCM), which will be most relevant in the sequel.

1.2.1 Gauge theory and boundary integrability

Chapters 2 and 3 are devoted to extending the results of [50, 51] to quantum integrable spin chains with boundary. They are largely based on the works [29, 30].

We begin chapter 2 by reviewing the boundary Yang-Baxter equation (BYBE), introduced by Cherednik in [39] and further expounded upon by Sklyanin in [152]. Solutions to the BYBE, known as K-matrices, determine integrable boundary conditions for quantum integrable spin chains. The BYBE also arises as a constraint on the scattering matrix of excitations off the boundary in a QIFT. It ensures that the factorisability of scattering processes continues to hold in the presence of the boundary. There is no established notion of a quasi-classical K-matrix, and so we must introduce our own definition. This leads to a boundary analogue of the classical Yang-Baxter equation (CYBE) satisfied by the order $\hbar$ contribution to a quasi-classical K-matrix. We also discuss a generalisation of the BYBE to include the possibility of degrees of freedom living at the boundary.

To generate quasi-classical K-matrices from CS_4 we study it on an $\mathbb{Z}_2$ orbifold. The $\mathbb{Z}_2$ acts as a reflection in the topological direction and simultaneously swaps the sign of the spectral parameter in the holomorphic direction. This must be lifted to the adjoint bundle of the gauge theory, which is achieved by specifying a symmetric pair. We show how configurations of Wilson lines on the orbifold can be viewed as living on a manifold with boundary. Quasi-classical K-matrices are generated as the

VEVs of reflecting Wilson lines. A distinguished set of line operators is identified which have support on the singular points of the orbifold. We refer to these as boundary line operators. In the context of quantum integrable spin chains these describe degrees of freedom living at the boundary. They must be included to obtain the full range of quasi-classical K-matrices, even those which apparently do not include boundary degrees of freedom.

By identifying a suitable orbifold propagator we calculate the order $\hbar$ contribution to a rational quasi-classical K-matrix with boundary degrees of freedom. This is novel, although it can be derived from relations appearing in the literature. We verify explicitly that it obeys the boundary analogue of the CYBE. Using a uniqueness result proved later, corresponding quasi-classical K-matrices in the fundamental representation are identified to all orders in $\hbar$.

The second half of chapter 2 focuses on the emergence of the twisted Yangian from CS_4. The twisted Yangian is the quantum group which governs rational solutions to the BYBE in the same way that the Yangian presides over rational solutions to the YBE. It is a coideal subalgebra of the Yangian, and in the context of QIFT can be interpreted as consisting of those Yangian charges which continue to be preserved in the presence of a boundary. Guided by [50], we study the OPE of bulk and boundary line operators. This corresponds to the coideal structure of the twisted Yangian, which is sufficient to determine its embedding into the Yangian. We infer that boundary line operators are labelled by representations of the twisted Yangian. The framing anomaly is also discussed, and we identify the corrected configuration leading to a K-matrix.

The RTT presentation of the twisted Yangian is then obtained from CS_4, affirming that boundary line operators are indeed labelled by its representations. We identify the gauge theory realisation of the Sklyanin determinant, the boundary analogue of the quantum determinant. Using this we prove that a rational K-matrix without boundary degrees of freedom in the fundamental representation is uniquely determined up to scale by its order $\hbar$ term. Generalisations of this proof are also discussed.

In chapter 3 some these results are extended to the trigonometric and elliptic cases. These involve a more complicated boundary condition and a non-trivial vacuum respectively, which restrict the possible lifts of the $\mathbb{Z}_2$ action to the adjoint bundle. We also find that there are more singular points on the orbifold, and hence there is greater freedom in introducing boundary line operators. By working in a holomorphic gauge we derive a general expression for the order $\hbar$ contribution to a quasi-classical K-matrix that applies in both the trigonometric and elliptic cases.

Concentrating on the trigonometric case, we classify all lifts of the $\mathbb{Z}_2$ action to the adjoint bundle using constant involutive automorphisms of $\mathfrak{g}$ which are compatible with the boundary conditions. For the simplest examples of these lifts we determine the order $\hbar$ contributions to the corresponding trigonometric K-matrices in the fundamental representation and compare with examples in the literature, finding agreement. We also propose lifting the $\mathbb{Z}_2$ action with more general bundle morphisms, and demonstrate that doing so leads to families of trigonometric K-matrices we would otherwise have missed.

In the elliptic case we classify all suitable lifts of the $\mathbb{Z}_2$ action to the adjoint bundle of the gauge theory. For all examples of these lifts we determine the order $\hbar$ contributions to the corresponding elliptic K-matrices in the fundamental representation. We then match these to examples appearing in the literature.

We explain why the proof of uniqueness for rational K-matrices does not extend straightforwardly trigonometric and elliptic cases. This prevents us from deducing all orders results.

1.2.2 Twistor actions for 4d integrable theories

Chapter 4 focuses on twistor actions for 4d integrable theories. Following a recent proposal of Costello [44] we study holomorphic Chern-Simons (HCS) theory on twistor space with a meromorphic measure. It is based on elements of the paper [31].

We begin by reviewing twistor theory in Euclidean signature as introduced by Penrose and Atiyah [141, 11]. The obstruction to studying HCS on twistor space is then discussed, namely, that there is no global holomorphic $(3, 0)$-form on twistor space. We then explain Costello's proposal to bypass this by using a meromorphic form and discuss how this leads to theories on spacetime admitting a 4d analogue of a Lax connection. The choice of meromorphic form on twistor space partially breaks Lorentz, and more generally conformal, invariance. For choices of $(3, 0)$-form which are nowhere vanishing we show that HCS descends to a spacetime theory which has classical equations of motion equivalent to the ASDYM equations.

Turning our attention to explicit examples of the meromorphic $(3, 0)$-form, we verify Costello's claim that if it has two double poles the resulting spacetime theory is the 4d WZW model (WZW_4) of [109, 130]. Its classical equation of motion is Yang's equation, which is equivalent to the ASDYM equations. A number of features of this theory are manifest from the twistor description. We then consider a $(3, 0)$-form with a single fourth order pole, leading to the cubic action of [104, 105, 139]. Its classical equations of motion are also equivalent to the ASDYM equations. The twistor description explains the symmetry of the theory under certain special conformal transformations and combined dilations and left handed rotations, which do not appear to have been spotted before. Finally we consider a $(3, 0)$-form with four distinct simple poles at which we impose trigonometric boundary conditions. This leads to an action which agrees off-shell with WZW_4, but nonetheless can be related to a proposed action for the ASDYM equations appearing in [124].

We also consider a $(3, 0)$-form which does have zeros, and find that it leads to a 4d theory of coupled σ-models. Its classical equations of motion are not equivalent to the ASDYM equations, although they can be interpreted as the zero-curvature equations for a suitable 4d analogue of a Lax connection. This example is interesting because in the next chapter we find that under symmetry reduction it leads to a 2d integrable theory of coupled σ-models which is not known to arise as a symmetry reduction of the ASDYM equations.

Finally we discuss reality conditions. We begin by explaining how to obtain 4d integrable theories in Lorentzian and utlrahyperbolic signatures starting with a 6d Chern-Simons theory (CS_6) on twistor correspondence space. This is important for two reasons: many of the 4d theories we obtain do not admit natural reality conditions in Euclidean signature and many lower dimensional integrable systems arise only as symmetry reductions in ultrahyperbolic signature. We also discuss how to introduce natural reality conditions on CS_6 and those they induce on the corresponding 4d integrable theory.

1.2.3 CS_4 as a symmetry reduction of CS_6 on twistor correspondence space

Chapter 5 proposes a connection between two rather different approaches to the study of classical integrable systems. It is also based on components of the paper [31].

On the one hand many integrable PDEs are known to arise as symmetry reductions of the ASDYM equations [162, 124], and on the other a plethora of 2d CIFTs have been obtained by introducing order and disorder defects in CS_4, and then eliminating the partial connection using its classical equations of motion [53]. There are two reasons that we might believe these approaches are connected. Firstly, they are both descriptions of 2d integrable models using non-abelian gauge theory, and secondly, they both involve geometrising the spectral parameter. In the case of the ASDYM equations the spectral parameter can be understood as a coordinate on twistor space.

To connect these we start with HCS on twistor space defined using a meromorphic $(3,0)$-form, or more generally CS_6 on twistor correspondence space. From the results of chapter 4, this describes a 4d integrable theory on spacetime. Quotienting by a subgroup of the conformal group with 2 dimensional orbits then leads to a 2d CIFT. If the $(3,0)$-form is nowhere vanishing, then this effectively implements a symmetry reduction of the ASDYM equations at the level of the action.

Alternatively we can lift our chosen subgroup of the conformal group to twistor space and perform the reduction there. This leads to CS_4 with insertions of disorder defects, and eliminating the partial connection leads to the 2d CIFT obtained by symmetry reduction.

This set up is illustrated in the diagram below:

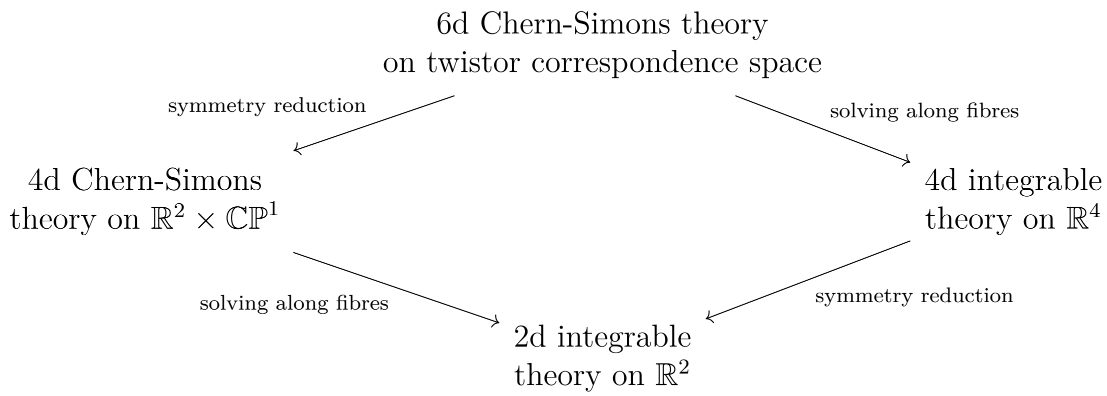


Fig. 1.2 A guide to the relationship between CS type theories and integrable systems in dimensions 2 & 4

We begin chapter 5 by verifying this proposal in the simplest possible case, where CS_6 describes WZW_4, and we quotient by a 2d group of translations with orbits on which the induced metric is non-degenerate. This leads to the PCM and its CS_4 description. We also discuss the result of applying this same reduction for different choices of $(3,0)$-form. Notably the 4d cubic action descends to a 2d CIFT which had not previously been obtained from CS_4, and the 4d coupled σ-models descend to a theory which is not known to arise as a symmetry reduction of the ASDYM equations.

We go on to perform the reduction of HCS on twistor space by a 1 dimensional group of translations. This necessitates reviewing the minitwistor correspondence introduced by Hitchin in [87]. The reduction gives a 5d Chern-Simons theory on minitwistor correspondence space with equations of motion related to the Bogomolny equations. We also briefly discuss performing the same reduction in ultrahyperbolic signature.

1.3 Background

In this section we define CS_4 and outline some of its connections to integrable models. It largely consists of a review of [50]. Elements of [51] also appear, and towards the end we briefly introduce some of the ideas in [53]. We refer the reader to these three papers and the review article [170] for a more comprehensive introduction.

1.3.1 Yang-Baxter equation

We begin by introducing the YBE.

As mentioned in the introduction, it plays two roles in the theory of quantum integrable systems. From the perspective of 2d QIFT, the YBE arises as a constraint on the S-matrix. Alternatively, its solutions can be used to construct the Hilbert space and conserved charges of a quantum integrable spin chain. The YBE is also important mathematically, having connections to abstract algebras known as infinite dimensional quantum groups. For a collection of some of the most important works pertaining to the YBE see [94]. An introduction to its role in statistical mechanics can be found in [19], and for a mathematical perspective see [35]. We will be interested in the YBE in its own right.

We refer to a solution of the YBE as an R-matrix. This is a linear map

$$R(z_1, z_2) : V_1 \otimes V_2 \to V_1 \otimes V_2 \,, \tag{1.1}$$

for V_1, V_2 complex vector spaces and z_1, z_2 complex spectral parameters on which it depends meromorphically. The YBE is the constraint:

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2) \,, \tag{1.2}$$

where the subscripts indicate on which factor of the tensor product $V_1 \otimes V_2 \otimes V_3$ the R-matrix acts. For example, $R_{12}(z_1, z_2)$ acts as $R(z_1, z_2)$ on the first two factors and as the identity on the third.

It is more easily understood geometrically, as illustrated in figure 1.3. In this diagram each line is associated with a vector space and complex spectral parameter. Whenever two lines cross we act on the tensor product of the corresponding vector spaces with the R-matrix, taking for its arguments the associated spectral parameters. The arrows indicate the order in which the R-matrices act. Equality of the two sides can readily be seen to give the YBE.

Naively, the YBE spectacularly over-determines the R-matrix. For example, in the simplest case that all three vector spaces are copies of the same V the YBE consists of $\mathcal{O}((\dim V)^6)$ equations whereas the R-matrix itself has only $\mathcal{O}((\dim V)^4)$ components. It is therefore surprising that the YBE admits many non-trivial solutions, and it is natural to ask if they have a common underlying origin.

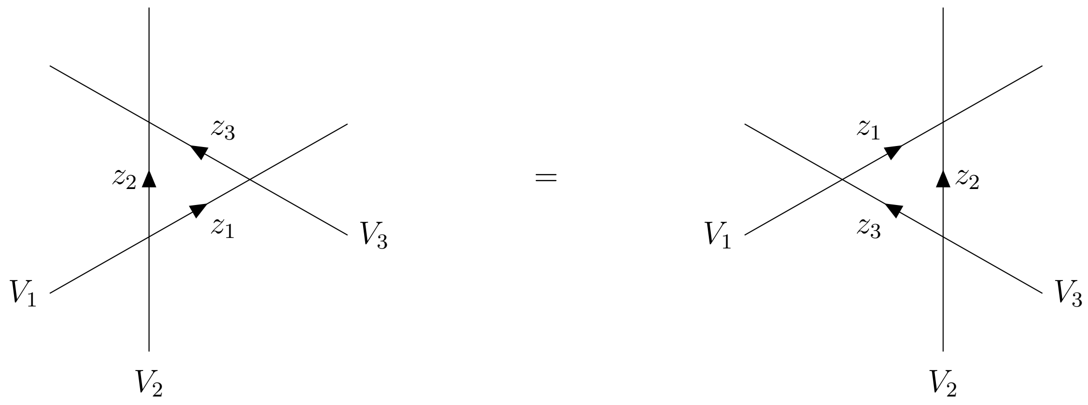


Fig. 1.3 Yang-Baxter equation

We will be interested in solutions to the YBE satisfying the following set of assumptions:

- The R-matrix depends only on the difference between the spectral parameters it takes as its arguments: $R(z_1, z_2) = R(z_1 - z_2)$. (In the context of 2d QIFT the role of the spectral parameter is played by the rapidity, and so this assumption follows from Lorentz invariance.)

- The R-matrix is a formal power series in a parameter $\hbar$ and equal to the identity in the $\hbar \to 0$ limit:

$$R(z) = \mathbb{1}_{V_1 \otimes V_2} + \hbar r_{V_1 \otimes V_2}(z) + \mathcal{O}(\hbar^2)\,. \tag{1.3}$$

  Here $r_{V_1 \otimes V_2}(z)$ is a meromorphic function of z with values in $\mathrm{End}(V_1 \otimes V_2)$, and is known as the classical r-matrix. R-matrices of this form are referred to as quasi-classical. Expanding the YBE as a formal power series in $\hbar$ we obtain a non-trivial constraint on the classical r-matrix at order $\hbar^2$. This reads

$$[r_{23}(z_{23}), r_{13}(z_{13})] + [r_{13}(z_{13}), r_{12}(z_{12})] + [r_{23}(z_{23}), r_{12}(z_{12})] = 0\,, \tag{1.4}$$

  where we have introduced the notation $z_{ij} = z_i - z_j$ and as above the subscripts indicate on which factors of the tensor product the r-matrix acts. The above equation is known as the classical Yang-Baxter equation (CYBE).

- Since the CYBE uses only the commutator on the vector spaces $\operatorname{End} V_i$, it continues to make sense for a classical r-matrix taking values in $\mathfrak{g} \otimes \mathfrak{g}$ for $\mathfrak{g}$ a complex Lie algebra. We can recover an r-matrix with values in $\operatorname{End}(V_1 \otimes V_2)$ by fixing representations ρ_{V_1}, ρ_{V_2} of $\mathfrak{g}$ and forming $r_{V_1 \otimes V_2}(z) = \rho_{V_1} \otimes \rho_{V_2}(r(z))$. We will be interested in quasi-classical R-matrices with corresponding classical r-matrices of this type.

Solutions to the CYBE for finite dimensional simple $\mathfrak{g}$ have been classified by Belavin and Drinfeld [22] under the assumption that the r-matrix is non-degenerate. They can be separated into three families distinguished by whether the classical r-matrix can be written in terms of rational, trigonometric or elliptic functions. We will refer to quasi-classical R-matrices as rational, trigonometric or elliptic according to the family of their corresponding classical r-matrix. We emphasise that in general a classical r-matrix for $\mathfrak{g}$ need not be $\mathfrak{g}$-invariant. Indeed, this is only ever true in the rational case.

For a given classical r-matrix and representations V_1, V_2 of $\mathfrak{g}$ there is not always a corresponding quasi-classical R-matrix. Only certain representations are permissible. For each family of classical r-matrices there is an associated infinite dimensional quantum group, which in the rational case is the Yangian. For an R-matrix to exist the representations V_1, V_2 must lift to the corresponding quantum group.

Since the YBE is homogeneous, its solutions are defined only up to multiplication by a function $F(z, \hbar)$. (Strictly F is a formal power series in $\hbar$ whose coefficients are meromorphic functions in z.) Given two R-matrices obeying

$$R_1(z) = F(z, \hbar) R_2(z) \tag{1.5}$$

for $F(z, \hbar) = 1 + \mathcal{O}(\hbar)$ we write $R_1(z) \sim R_2(z)$. This is an equivalence relation and we refer to its equivalence classes as R-matrices up to scale. A representative in each equivalence class can be specified by imposing further constraints on the R-matrix.

Example 1.3.1. It is useful to have an example of a quasi-classical R-matrix to illustrate some of the ideas above. For general simple $\mathfrak{g}$ a rational solution to the CYBE is given by

$$r(z) = \frac{c}{z} \tag{1.6}$$

for c the split quadratic Casimir of $\mathfrak{g}$ dual to the bilinear tr.[1] Given this classical r-matrix is homogeneous it's natural to require that the corresponding quasi-classical R-matrix depends on z only through the combination $\hbar/z$. Taking $U = V_1 = V_2$ to be the fundamental representation of $\mathfrak{sl}_n(\mathbb{C})$ we find that

$$r_{U\otimes U}(z) = \frac{1}{z}\left(\mathbb{P}_{U\otimes U} - \frac{1}{n}\mathbb{1}_{U\otimes U}\right), \tag{1.7}$$

where $\mathbb{P}_{U\otimes U}$ is the permutation operator exchanging the two factors of the tensor product. The fundamental representation of $\mathfrak{sl}_n(\mathbb{C})$ lifts to the Yangian, and so the corresponding quasi-classical R-matrix exists and is given by

$$R(z) = F(z,\hbar)\left(\mathbb{1}_{U\otimes U} + \frac{\hbar}{z}\mathbb{P}_{U\otimes U}\right) \sim \mathbb{1}_{U\otimes U} + \frac{\hbar}{z}\mathbb{P}_{U\otimes U}\,. \tag{1.8}$$

Here

$$F(z,\hbar) = 1 - \frac{\hbar}{nz} + \mathcal{O}\left(\frac{\hbar^2}{z^2}\right). \tag{1.9}$$

The higher order terms in F, and hence the overall scale of the R-matrix, can be fixed by requiring that it has unit quantum determinant,

$$\varepsilon_{i_1\dots i_n} R^{i_1}{}_{1,U}(z) R^{i_2}{}_{2,U}(z+\hbar)\dots R^{i_n}{}_{n,U}(z+(n-1)\hbar) = \mathbb{1}_U\,, \tag{1.10}$$

in both its factors. Here we have made the indices on the first factor of the R-matrix explicit.

This rational solution to the YBE is known as Yang's rational R-matrix [175]. For $n = 2$ it describes the XXX Heisenberg spin chain.

1.3.2 4d Chern-Simons theory

Here we introduce CS$_4$, closely following [50, 53].

CS$_4$ is defined on the product of 2d smooth manifold Σ and a Riemann surface C which is required to admit a closed, meromorphic 1-form ω. We write $\mathcal{V} = \Sigma \times C$, and note that we have a trivial double fibration $\Sigma \xleftarrow{\pi_\Sigma} \mathcal{V} \xrightarrow{\pi_C} C$. The dynamical field is a connection on a principal G-bundle over $\mathcal{V}$ for G a connected complex Lie group with reductive complex Lie algebra $\mathfrak{g}$. Here reductive means that $\mathfrak{g}$ is the direct sum of a

[1] This is not quite the most general rational solution to the CYBE, see [153] for a discussion. The broader family of solutions lead to deformed Yangians [154], which would be interesting to obtain from CS$_4$.

semisimple and abelian Lie algebra. Assuming that this principal G-bundle is trivial, the connection can be represented by a 1-form with values in $\mathfrak{g}$ which we denote by A. The action of CS_4 is

$$S_{\mathrm{CS}_4}[A] = \frac{i}{4\pi}\int_{\mathcal{V}} \omega \wedge \mathrm{CS}(A) \tag{1.11}$$

where abuse notation by writing ω for its pullback to $\mathcal{V}$ by π_C.

$$\mathrm{CS}(A) = \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\right) \tag{1.12}$$

is the CS 3-form for tr a choice of $\mathfrak{g}$-invariant symmetric bilinear on $\mathfrak{g}$. In many cases of interest $\mathfrak{g}$ is simple, in which case we take tr to be the minimal $\mathfrak{g}$-invariant bilinear. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ this coincides with the trace in the fundamental representation, and more generally $\mathrm{tr} = \kappa/2\mathbf{h}^\vee$ for κ the Killing form and $\mathbf{h}^\vee$ the dual Coxeter number.

CS_4 is a non-abelian gauge theory in the usual sense: we identify field configurations related by automorphisms of the G-bundle. For a trivial G-bundle such automorphisms can be represented by maps $g : \mathcal{V} \to G$ which act on A by

$$A \mapsto g^{-1}\,\mathrm{d}g + g^{-1} A g\,. \tag{1.13}$$

Although the action is invariant under infinitesimal gauge transformations of this form, it picks up a WZW type term under finite transformations. Since we will be working either classically or perturbatively in $\hbar$ we can safely ignore this failure of gauge invariance. Nonetheless, non-perturbative definitions of CS_4 have recently been given in [9, 52] using brane constructions.

There is an obvious redundancy in A under

$$A \mapsto A + \omega\xi \tag{1.14}$$

for $\xi \in C^\infty(\mathcal{V})$, so that $A \in \langle\omega\rangle_{C^\infty(\mathcal{V})}\backslash\Omega^1(\mathcal{V})$. This redundancy can be eliminated by requiring that the $(1,0)$-component of A in the C direction vanishes. Fixing coordinates (x, y) on Σ and holomorphic coordinates $(z, \bar{z})$ on C we have

$$A = A_\Sigma + \bar{A}_C = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_{\bar{z}}\,\mathrm{d}\bar{z}\,. \tag{1.15}$$

We then refer to A as a partial connection. Having eliminated the ambiguity in this way we must replace the exterior derivative appearing in the gauge transformation (1.13) with

$$\mathrm{d}' = \mathrm{d}_\Sigma + \bar{\partial}_C = \mathrm{d}x\partial_x + \mathrm{d}y\partial_y + \mathrm{d}\bar{z}\partial_{\bar{z}}\,. \tag{1.16}$$

We can also replace d by d′ in the action because the ∂_C component drops out upon wedging with ω. It is clear that the action of CS_4 is sensitive to only the topology of Σ, but depends on the complex structure of C through ω. More precisely, it is invariant under diffeomorphisms of $\mathcal{V}$ which preserve ω. Two subgroups are of particular interest:

- Diffeomorphisms which pushforward trivially to C. These are generated by vector fields

$$\xi = \xi^x \partial_x + \xi^y \partial_y \tag{1.17}$$

  where ξ^x, ξ^y are smooth functions on $\mathcal{V}$. We will often refer to elements of this subgroup as diffeomorphisms in the Σ direction.

- Diffeomorphisms which pushforward trivially to Σ. These exist if ω is nowhere vanishing, so that ω^{-1} is well defined and generates a global symmetry of C preserving ω. This only occurs for $(C,\omega) = (\mathbb{C}, \mathrm{d}z)$, $(\mathbb{C}/2\pi i\mathbb{Z}, \mathrm{d}z)$ and $(\mathrm{T}^2_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}), \mathrm{d}z)$. We will later see that these three choices correspond to rational, trigonometric and elliptic R-matrices respectively.

The action also has a global G symmetry acting by constant gauge transformations. Both this and diffeomorphism invariance in the Σ direction can be broken by the boundary conditions we impose on A. Such boundary conditions can arise at, e.g., poles in ω.

The classical equations of motion of CS_4 are

$$\omega \wedge F(A) = 0\,. \tag{1.18}$$

In a gauge where $\bar{A}_C = 0$ these imply that A_Σ is a flat connection on Σ varying holomorphically in C. In general this gauge will be obstructed by the boundary conditions on A.

1.3.3 Rational R-matrices from CS_4

In this subsection we review how solutions to the YBE are generated as the VEVs of crossing Wilson lines in CS_4, as originally argued in [42, 50].

Quantum CS_4

We treat CS_4 as a perturbative quantum field theory, which essentially means that we define the path integral using the Feynman diagram expansion. Writing $\mathcal{A}$ for the affine space of partial connections on the G-bundle and $\mathcal{G}$ for the group of gauge transformations, the VEV of an observable $\mathcal{O} : \mathcal{A}/\mathcal{G} \to \mathbb{C}$ is heuristically defined by

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, \mathcal{O}[A] \exp(-S_{\mathrm{CS}_4}[A]/\hbar)}{\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, \exp(-S_{\mathrm{CS}_4}[A]/\hbar)}\,. \tag{1.19}$$

In order to make sense of this we will need to expand around an isolated solution to the classical equations of motion modulo gauge transformations, and also fix the gauge.

Naive power counting suggests the CS_4 should be non-renormalizable. Remarkably, however, Costello proved in [42] that the theory is renormalizable by using a Lorenz type gauge fixing. This was based on the rather intricate BV techniques developed in [41, 46]. In brief, after gauge fixing all BRST invariant local operators vanish on-shell and so any counterterm can be removed by a field redefinition. The apparent non-renormalizability plays an important role, however: it implies that the theory is IR free which leads to a local prescription for evaluating correlation functions.

Following [50], we restrict (C, ω) to one of the three cases $(\mathbb{C}, \mathrm{d}z)$, $(\mathbb{C}/2\pi i\mathbb{Z}, \mathrm{d}z)$ and $(\mathrm{T}^2_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \mathrm{d}z)$. These three choices lead to rational, trigonometric and elliptic solutions of the YBE respectively. In each case it's possible to choose boundary conditions so that full diffeomorphism invariance in the Σ direction is preserved. In this subsection we take $(C, \omega) = (\mathbb{C}, \mathrm{d}z)$, which is by far the simplest of the three possibilities, leaving the trigonometric and elliptic cases for subsection 1.3.5.

Our boundary conditions are that A should vanish to first order at $z = \infty$. This ensures that $\mathrm{CS}(A)$ has a second order zero at $z = \infty$ compensating the double pole in ω. When varying the action there is therefore no overall pole at $z = \infty$, and so no boundary terms are generated. Infinitesimal gauge transformations are also required to vanish at infinity for consistency. (Both the gauge field and infinitesimal gauge transformations tend to 0 at infinity in Σ if it is non-compact.) We expand around the vacuum $A = 0$ which is easily verified to be isolated.

As mentioned in subsection 1.3.2, in the rational case $\omega^{-1} = \partial_z$ generates a translation symmetry in $C = \mathbb{C}$. There are also two further simplifications. The first is that global G-invariance of the theory is unbroken by the vacuum or boundary conditions, and the second is that the partition function is invariant under simultaneous rescalings $z \mapsto \lambda z$, $\hbar \mapsto \lambda\hbar$. (The latter fails in the elliptic and trigonometric cases since z is periodic.)

Operators

At the classical level the simplest class of observables in CS_4 are Wilson lines

$$\mathcal{W}_V[\gamma, A] = \mathrm{P}\exp\left(\int_\gamma A_{\Sigma,V}\right) \tag{1.20}$$

along a curve $\gamma \subset \mathcal{V}$, for V a representation of $\mathfrak{g}$.[2] Since we only have a partial connection, Wilson lines along curves varying in C are not well defined. We therefore require γ to be supported at a point $z \in C$. For Σ non-compact Wilson lines extending to infinity are permitted. Indeed, our boundary conditions ensure that they are gauge invariant without taking a trace.

There is in fact a larger class of line operators which can be introduced in CS_4. To build them we must first introduce $\mathfrak{g}[[u]]$, the Lie algebra of power series in the formal parameter u with coefficients in $\mathfrak{g}$. Its elements can be expressed as

$$X = X^{(0)} + X^{(1)}u + X^{(2)}u^2 + \cdots \in \mathfrak{g}[[u]] \tag{1.22}$$

for $X^{(k)} \in \mathfrak{g}$. The sum need not terminate after a finite number of terms. Fixing a basis $\{t_a\}_{a=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$,

$$\{t_a^{(n)} = t_a u^n\}_{a=1, n\in\mathbb{Z}_0^+}^{\dim\mathfrak{g}} \tag{1.23}$$

is a basis of $\mathfrak{g}[[u]]$. Employing summation convention, the Lie algebra structure is then determined by

$$[t_a^{(m)}, t_b^{(n)}] = f_{ab}{}^c t_c^{(m+n)}\,, \tag{1.24}$$

where $f_{ab}{}^c$ are the structure constants of $\mathfrak{g}$. The algebra is $\mathbb{Z}_0^+$ graded with $\deg t_a^{(n)} = n$. If $\mathfrak{g}$ is semisimple then $\mathfrak{g}[[u]]$ is generated as a Lie algebra by $\{t_a^{(0)}, t_a^{(1)}\}_{a=1}^{\dim\mathfrak{g}}$ obeying

$$\begin{gathered}[t_a^{(0)}, t_b^{(0)}] = f_{ab}{}^c t_c^{(0)}\,, \qquad [t_a^{(0)}, t_b^{(1)}] = f_{ab}{}^c t_c^{(1)}\,, \\ [t_a^{(1)}, [t_b^{(1)}, t_c^{(1)}]] + [t_b^{(1)}, [t_c^{(1)}, t_a^{(1)}]] + [t_c^{(1)}, [t_a^{(1)}, t_b^{(1)}]] = 0\,.\end{gathered} \tag{1.25}$$

The first condition tells us that $\{t_a^{(0)}\}_{a=1}^{\dim\mathfrak{g}}$ generates a subalgebra isomorphic to $\mathfrak{g}$, and the second that the $t_a^{(1)}$ transform in the adjoint with respect to this subalgebra. The

[2]Following the convention used in [50], Wilson lines are defined by

$$(\mathrm{d}_\Sigma + A_\Sigma)\mathcal{W}_V[\gamma(v), A]^{-1} = 0\,. \tag{1.21}$$

This means that the plus sign in the definition of the Wilson line is correct, but the convention for path ordering is from left to right. This makes the connection to the integrable systems literature as clear as possible.

final condition is equivalent to

$$[t_c^{(1)}, t_d^{(1)}]\Phi^{cd}{}_{ab} = 0 \tag{1.26}$$

for $\Phi \in \mathrm{End}(\wedge^2\mathfrak{g})$ the orthogonal projection onto the kernel of $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$. Since $\mathfrak{g}[[u]]$ is infinite dimensional we should define its representations with a little care: we require that there exists an $N \in \mathbb{Z}^+$ such that for all $n \geq N$ the image of $t_a^{(n)}$ vanishes. This ensures that the image of a general $X \in \mathfrak{g}[[u]]$ is well defined.

We can then introduce

$$A_\Sigma^{\mathfrak{g}[[u]]} = \sum_{m=0}^{\infty} \frac{1}{m!} t_a^{(m)} \partial_z^m A_\Sigma^a \tag{1.27}$$

which we view as a 1-form with values in $\mathfrak{g}[[u]]$. Fixing a representation V of $\mathfrak{g}[[u]]$ we can define a generalised line operator

$$\mathcal{W}_V[\gamma, A] = \mathrm{P}\exp\left(\int_\gamma A_{\Sigma,V}^{\mathfrak{g}[[u]]}\right) = \mathrm{P}\exp\left(\int_\gamma \sum_{m=0}^{\infty} \frac{1}{m!} t_{a,V}^{(m)} \partial_z^m A_\Sigma^a\right). \tag{1.28}$$

As for ordinary Wilson lines the curve γ must be supported at a point $z \in C$. Note that for the above to be consistent with invariance under simultaneous rescalings of z and $\hbar$ we should view the $t_a^{(m)}$ as being of order $\hbar^m$. Compatibility with gauge symmetry requires that we interpret infinitesimal gauge transformations similarly

$$\varepsilon^{\mathfrak{g}[[u]]} = \sum_{m=0}^{\infty} \frac{1}{m!} t_a^{(m)} \partial_z^m \varepsilon^a . \tag{1.29}$$

In quantum CS_4 we will be interested in computing the VEVs of configurations of Wilson lines. To define the path integral of CS_4 we first need to fix a gauge, which is achieved by specifying a metric on Σ. IR freedom of CS_4 after gauge fixing ensures that as we scale up this metric, effectively moving observables apart in Σ, interactions become weaker. Given two Wilson lines which do not cross, we can scale up the metric in this way to make the contribution from gluon exchange to their VEV arbitrarily small. Similarly, given a pair of crossing Wilson lines we can scale up the metric so that the only contributions to their VEV arise from gluon exchanges in an arbitrarily small neighbourhood of their crossing in Σ. In this way we obtain a local prescription for evaluating the VEVs of configurations of Wilson lines. The basic building block is the VEV of crossing Wilson lines, illustrated in figure 1.4.

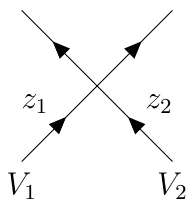


Fig. 1.4 R-matrix from CS_4

This yields a linear map

$$R(z_1 - z_2) : V_1 \otimes V_2 \to V_1 \otimes V_2 \tag{1.30}$$

depending on the difference of the spectral parameters, a consequence of the translational invariance of the theory in C. It is holomorphic away from $z_1 = z_2$, which follows from the formal properties of the path integral. We will see this in explicit examples shortly.

Notice that diffeomorphism invariance in the Σ direction ensures that the VEVs of the following two configurations of Wilson lines are the same.

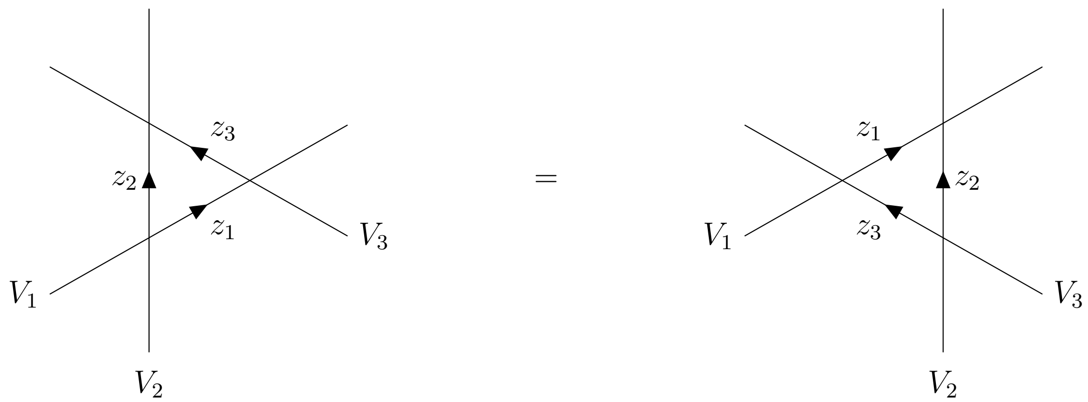


Fig. 1.5 YBE in CS_4

The YBE for $R(z)$ follows. In this way CS_4 generates quasi-classical R-matrices.

A crucial feature of CS_4 is that 2-loop anomalies to line operators mean that they are labelled by representations of the Yangian of $\mathfrak{g}$, rather than $\mathfrak{g}[[u]]$. An ordinary Wilson line in a representation V of $\mathfrak{g}$ is permitted if and only if V lifts to a representation of the corresponding Yangian. In addition, if γ curves in Σ with respect to some reference framing, then in the quantum theory $\mathcal{W}_V[\gamma, A]$ is anomalous and requires special treatment. This actually necessitates a slight modification of the above realisation of

R-matrices in CS_4. We shall review how to detect these anomalies and some of their consequences in subsection 1.3.4.

Propagator

In order to make progress studying CS_4 as a quantum theory we need to identify a suitable propagator. This is possible in the rational case on account of the extra symmetries present. As discussed above, CS_4 is local in Σ and so we can without loss of generality take $\Sigma = \mathbb{R}^2$. We make this choice now.

To determine a propagator we must first fix the gauge. This is achieved by specifying a metric on Σ, which we take to be the standard flat metric δ_Σ. Note C has a canonical metric $\omega\bar{\omega}$, and combining them gives

$$g = \delta_\Sigma + \frac{1}{2}\omega\bar{\omega} = \mathrm{d}x^2 + \mathrm{d}y^2 + \frac{1}{2}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{1.31}$$

the standard flat metric on $\mathcal{V}$. We then impose the Lorenz type gauge

$$*\,\mathrm{d} * A = \partial_x A_x + \partial_y A_y + 4\partial_z A_{\bar{z}} = D^i A_i = 0\,. \tag{1.32}$$

Here the indices $i, j, k, \ldots$ take values in $\{x, y, \bar{z}\}$, and $D^i = (\partial_x, \partial_y, 4\partial_z)$. Gauge fixing using the Faddeev-Popov procedure gives

$$\begin{aligned} &S_{\text{g.f.}}[A, h, c, \tilde{c}] \\ &\quad= \frac{1}{\pi}\int_{\mathcal{V}} \mathrm{vol}_g\, \mathrm{tr}\Big(\varepsilon^{ijk} A_i \partial_j A_k + \frac{2}{3}\varepsilon^{ijk} A_i A_j A_k + 2h D^i A_i + 2\tilde{c} D^i(\partial_i c + [A_i, c])\Big)\,, \end{aligned} \tag{1.33}$$

where h, c and $\tilde{c}$ are the Nakanishi-Lautrup field, ghost and anti-ghost respectively. The epsilon tensor is normalized by $\varepsilon^{xy\bar{z}} = 1$, and

$$\mathrm{vol}_g = \frac{i}{4}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{1.34}$$

In practice we will only need the components of the propagator involving the partial connection A, defined by

$$\hbar P_{ij}(v_1 - v_2) = \langle A_i(v_1) \otimes A_j(v_2)\rangle_0 \tag{1.35}$$

where the subscript 0 on the expectation on the right hand side indicates that we use only the quadratic part of the action in the path integral.

The propagator is the Green's function for the classical equation of motion, and so obeys

$$\varepsilon^{ijk}\partial_j P_{k\ell}(v) - D^i Q_\ell(v) = \frac{c\pi}{2}\delta^i{}_\ell \delta^4_g(v)\,, \qquad D^i P_{ij}(v) = 0\,. \tag{1.36}$$

Here $c = \mathrm{tr}^{-1} \in \mathfrak{g}\otimes\mathfrak{g}$ is the split Casimir and the δ-function is normalised so that

$$\int_{\mathcal{V}} \mathrm{vol}_g\, \delta^4_g(v) = 1\,. \tag{1.37}$$

We have also introduced the component of the propagator involving the Nakanishi-Lautrup field h

$$\hbar Q_i(v_1 - v_2) = \langle h(v_1)\otimes A_i(v_2)\rangle_0 \tag{1.38}$$

which in turn must obey

$$\hbar D^i Q_i(v) = -\frac{c\pi}{2}\delta^4_g(v)\,. \tag{1.39}$$

Clearly we can extract the Lie theoretic factor from the propagator by writing $P_{ij}(v) = c\Delta_{ij}(v)$. This is essentially a consequence of the global G-symmetry of the theory. We can solve the second equation in (1.36) by letting

$$\Delta_{ij}(v) = \pi\varepsilon_{ijk}D^k\mathcal{G}(v)\,. \tag{1.40}$$

The first is then solved by

$$\partial_i D^i \mathcal{G}(v) = \frac{1}{2}\delta^4_g(v) \tag{1.41}$$

if $Q_i = -\pi c\partial_i\mathcal{G}(v)$, which is also consistent with equation (1.39). The combination $\partial_i D^i$ appearing on the left hand side of equation (1.41) is simply the Laplace operator on $\mathcal{V} = \mathbb{R}^2\times\mathbb{C}\cong\mathbb{R}^4$ with Green's function

$$\mathcal{G}(v) = -\frac{1}{4\pi^2\|v\|^2} \tag{1.42}$$

for $\|v\|^2 = x^2 + y^2 + |z|^2$.[3] The components of the propagator involving A are therefore

$$P_{ij}(v) = -\frac{c}{4\pi}\varepsilon_{ijk}D^k\left(\frac{1}{\|v\|^2}\right)\,. \tag{1.43}$$

This is the form of the propagator used in [50].

[3]The conscientious reader may notice that the our definition of $\|v\|^2$ differs slightly from the choice suggested by the metric (1.31). This issue can be explained by noting that $\partial_i D^i$ does not exactly coincide with the Laplacian induced by g. This is because on a Kähler manifold $\Delta_{\mathrm{d}} = 2\Delta_{\bar\partial}$.

Computing the classical r-matrix

We have seen that R-matrices are generated as the VEVs of crossing Wilson lines in CS_4. This can be expressed as

$$R(z) = \langle \mathcal{W}_{V_1}[\gamma_1, A] \otimes \mathcal{W}_{V_2}[\gamma_2, A] \rangle \,, \tag{1.44}$$

where γ_1, γ_2 are (non-parallel) straight lines in Σ supported at the points $z, 0 \in C$. The classical limit can be identified by evaluating the observable on the vacuum configuration $A = 0$. This gives

$$R(z) = \mathbb{1}_{V_1 \otimes V_2} + \hbar r_{V_1 \otimes V_2}(z) + \mathcal{O}(\hbar^2) \,. \tag{1.45}$$

Here we have identified the term at order $\hbar$ with the classical r-matrix introduced in section 1.3.1. We expect that it should solve the CYBE, given in equation (1.4).

Having identified a suitable propagator we can compute the classical r-matrix explicitly. The only Feynman diagram contributing to the VEV of crossing Wilson lines at first order in $\hbar$ is illustrated below.

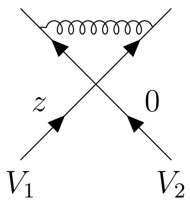


Fig. 1.6 Classical r-matrix from CS_4

By integrating the propagator over $\gamma_1 \times \gamma_2$, the contribution of this diagram is easily evaluated to be

$$r_{V_1 \otimes V_2}(z) = \frac{c_{V_1 \otimes V_2}}{z} \tag{1.46}$$

for $c_{V_1 \otimes V_2}$ the image of the split Casimir in the tensor product of the representations V_1 and V_2. Abstracting away the representations we obtain

$$r(z) = \frac{c}{z} \,, \tag{1.47}$$

the standard rational r-matrix appearing in [21, 22].

Now let us fix $\mathfrak{g}$ to be simple with tr the minimal $\mathfrak{g}$-invariant bilinear, and choose both Wilson lines to be in the fundamental representation $U = V_1 = V_2$. By fundamental representation we mean the defining vector representation for the classical simple Lie algebras, and the fundamental representation with smallest dimension in the exceptional cases. (For $\mathfrak{g} = \mathfrak{e}_6$ there are two such representations, either of which will do. For reasons we shall discuss shortly, Wilson lines in the 248 dimensional representation of $\mathfrak{e}_8$ do not exist in CS$_4$, so we must exclude this case.) It is proven in [51] that there is a unique rational R-matrix up to scale with classical r-matrix $r_{U\otimes U}(z)$. (This also follows from an earlier result of Drinfel'd [67].) This allows us to deduce the following results which hold to all orders in $\hbar$.

Example 1.3.2. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$

$$R(z) \sim \mathbb{1}_{U\otimes U} + \frac{\hbar}{z}\mathbb{P}_{U\otimes U}\,, \tag{1.48}$$

where we recall that $\mathbb{P}_{U\otimes U}$ is the permutation operator exchanging the factors of the tensor product. (Remember also that $\sim$ indicates agreement up to multiplication by $F(z,\hbar) = 1 + \mathcal{O}(\hbar)$ a formal power series in $\hbar$ with values in the ring of meromorphic functions of z.) This is Yang's rational R-matrix [175] which we previously introduced in example 1.3.1.

Example 1.3.3. For $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$

$$R(z) \sim \mathbb{1}_{U\otimes U} + \frac{\hbar}{z}\mathbb{P}_{U\otimes U} - \frac{\hbar}{z + \hbar(n-2)/2}\mathbb{Q}_{U\otimes U}\,, \tag{1.49}$$

where $\mathbb{P}_{U\otimes U}$ is as above and $\mathbb{Q}_{U\otimes U} = \delta^{-1}\otimes\delta$ for $\delta \in S^2U^\vee$ the symmetric bilinear used to define $\mathfrak{so}_n(\mathbb{C})$. This R-matrix was first introduced in [179].

Similar results can be deduced for the remaining simple Lie algebras [27, 150, 135]. In the above two examples the overall scales of the R-matrices are undetermined, but we will see in subsection 1.3.4 how they are also fixed in CS$_4$.

1.3.4 Yangians from CS$_4$

We now review how the Yangian, in both the RTT and J-presentations, emerges from CS$_4$ theory on $\mathcal{V}$. The framing anomaly is also discussed. Again, these were originally obtained in [42, 50].

OPE and Yangian

Consider two bulk Wilson lines in representations V_1 and V_2 of $\mathfrak{g}$, both supported at the same point $z \in \mathbb{C}$. Then suppose we bring them together in parallel. This forms a

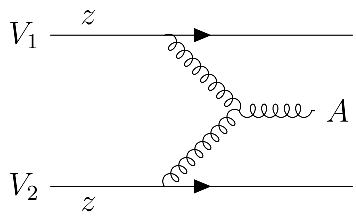


Fig. 1.7 Quantum correction to the bulk OPE

new line operator, also supported at $z \in \mathbb{C}$. Classically, this is simply a Wilson line in the tensor product representation $V_1 \otimes V_2$. However, in the quantum theory it receives a correction. At $\mathcal{O}(\hbar)$ this arises from the Feynman diagram illustrated in figure 1.7.

This was evaluated in [50] to give a contribution

$$-\frac{\hbar}{2} f_a{}^{bc} t_{b,V_1} \otimes t_{c,V_2} \int_\gamma \partial_z A^a_\Sigma \,. \tag{1.50}$$

The presence of a z derivative of A indicates that the result is not an ordinary bulk Wilson line. Instead, we've generated one of the generalised line operators introduced in subsection 1.3.3. To obtain an operation which is closed we therefore should compute the OPE of two of these generalised line operators. At first order in $\hbar$ the only new feature is that the external gauge field can couple directly to either $t^{(1)}_{a,V_1}$ or $t^{(1)}_{b,V_2}$. (Note here that $t^{(1)}_a$ is of order $\hbar$.) The OPE is therefore a generalised line operator with

$$\begin{aligned} t^{(0)}_{a,V_1\otimes V_2} &= t^{(0)}_{a,V_1} \otimes \mathbb{1}_{V_2} + \mathbb{1}_{V_1} \otimes t^{(0)}_{a,V_2} \,, \\ t^{(1)}_{a,V_1\otimes V_2} &= t^{(1)}_{a,V_1} \otimes \mathbb{1}_{V_2} + \mathbb{1}_{V_1} \otimes t^{(1)}_{a,V_2} - \frac{\hbar}{2} f_a{}^{bc} t^{(0)}_{a,V_1} \otimes t^{(0)}_{b,V_2} \,. \end{aligned} \tag{1.51}$$

(We continue to write $V_1 \otimes V_2$ for the representation of the resulting boundary line operator.)

To understand this we first need to reinterpret the classical OPE in a more abstract way. It takes two line operators in representations V_1 and V_2 of $\mathfrak{g}[[u]]$ and returns a line operator in the tensor product $V_1 \otimes V_2$, defined by (1.51) with $\hbar = 0$. Abstractly, the tensor product of representations is induced by the coproduct on $U(\mathfrak{g}[[u]])$, i.e., the morphism of associative algebras

$$\begin{aligned} \Delta_0 : U(\mathfrak{g}[[u]]) &\to U(\mathfrak{g}[[u]]) \otimes U(\mathfrak{g}[[u]]) \,, \\ t^{(m)}_a &\mapsto t^{(m)}_a \otimes 1 + 1 \otimes t^{(m)}_a \,. \end{aligned} \tag{1.52}$$

Here it is essential that we use the universal enveloping algebra (UEA) $U(\mathfrak{g}[[u]])$ rather than the Lie algebra itself since $\mathfrak{g}[[u]]$ does not contain a unit and $\mathfrak{g}[[u]]^{\otimes 2}$ does not admit a canonical Lie algebra structure. Note that the representation theory of a Lie algebra and its UEA are isomorphic. The representation $\rho_{V_1 \otimes V_2}$ is then simply the composition of $\rho_{V_1} \otimes \rho_{V_2}$ with Δ_0.

The quantum correction to the OPE deforms this coproduct structure to

$$\begin{aligned} \Delta_\hbar : t_a^{(0)} &\mapsto t_a^{(0)} \otimes 1 + 1 \otimes t_a^{(0)} \,, \\ t_a^{(1)} &\mapsto t_a^{(1)} \otimes 1 + 1 \otimes t_a^{(1)} - \frac{\hbar}{2} f_a{}^{bc} t_b^{(0)} \otimes t_c^{(0)} \,. \end{aligned} \tag{1.53}$$

However, this is not the whole story. Since the coproduct is a morphism of associative algebras, a deformation to it signals a deformation of the algebra structure on $U(\mathfrak{g}[[u]])$. The algebra compatible with the coproduct (1.53) is the Yangian, $\mathcal{Y}(\mathfrak{g})$. It is generated as an associative algebra by $\{t_a^{(0)}, t_a^{(1)}\}_{a=1}^{\dim \mathfrak{g}}$ obeying

$$\begin{gathered} [t_a^{(0)}, t_b^{(0)}] = f_{ab}{}^c t_c^{(0)} \,, \qquad [t_a^{(0)}, t_b^{(1)}] = f_{ab}{}^c t_c^{(1)} \,, \\ f_{ab}{}^d [t_c^{(1)}, t_d^{(1)}] + f_{ca}{}^d [t_b^{(1)}, t_d^{(1)}] + f_{bc}{}^d [t_a^{(1)}, t_d^{(1)}] = \hbar^2 Q_{abc}(t_\bullet^{(0)}) \,, \end{gathered} \tag{1.54}$$

for

$$Q_{abc}(t_\bullet^{(0)}) = \frac{1}{4} f_{ad}{}^g f_{be}{}^h f_{cf}{}^i f^{def} \{t_g^{(0)}, t_h^{(0)}, t_i^{(0)}\} \,. \tag{1.55}$$

Here we have introduced the notation

$$\{X_1, X_2, X_3\} = \frac{1}{|S_3|} \sum_{\sigma \in S_3} X_{\sigma(1)} X_{\sigma(2)} X_{\sigma(3)} \,. \tag{1.56}$$

This definition of the Yangian is known as Drinfeld's J-presentation [68], and the third equation in (1.54) is known as the Drinfeld terrific relation.[4] There exist alternative isomorphic definitions, one of which will be discussed shortly. Note that setting $\hbar = 0$ in (1.54) gives a presentation of $U(\mathfrak{g}[[u]])$. (See equation (1.25) for the analogous presentation of $\mathfrak{g}[[u]]$.)

The implication of this is that line operators are not labelled by representations of $\mathfrak{g}[[u]]$, but instead by representations of the Yangian $\mathcal{Y}(\mathfrak{g})$. One very important consequence of this is that not all ordinary Wilson lines associated to irreducible representations of $\mathfrak{g}$ exist in CS_4. In particular, for a given set of level 0 generators $\{t_{a,V}^{(0)}\}$ there may be no way to choose the $t_{a,V}^{(1)}$ so that they are compatible with

[4]For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ the terrific relation follows from the first two equations in (1.54), and so it must be replaced with a complicated condition. See chapter 12 of [35] for details.

the relations (1.54). Fortunately the representation theory of the Yangian is well understood, and many representations do lift [35, 50]. One important exception is the fundamental representation of $\mathfrak{e}_8$, which happens to coincide with the adjoint. This will prevent us from obtaining an RTT presentation for $\mathcal{Y}(\mathfrak{e}_8)$.

Whilst the order $\hbar$ deformation to the OPE is sufficient to guarantee that line operators are labelled by representations of the Yangian [68, 42, 43], it is also possible to derive the relations (1.54) directly. Here we will sketch the main ideas behind the calculation, which can be found in full in section 8 of [50].

The relations between the coefficients $t^{(m)}_{a,V}$ defining a generalised line operator are a consequence of gauge invariance, or more precisely BRST invariance after gauge fixing. In the quantum theory there are corrections to the infinitesimal BRST variation of a generalised line operator which cannot be eliminated through the introduction counterterms. Usually these would signal that the operator is not BRST closed and so does not exist in the quantum theory. In CS_4, however, they can be interpreted as deforming the algebra of the $t^{(m)}_{a,V}$ so that they determine a representation of $\mathcal{Y}(\mathfrak{g})$.

The first two equations in (1.54) are independent of $\hbar$, and follow from classical BRST invariance. The deformation appears only in the terrific relation. To see how it arises from an anomaly we rewrite it as

$$\Big([t^{(1)}_c, t^{(1)}_d] - \hbar^2 R_{cd}(t^{(0)}_\bullet)\Big)\Phi^{cd}{}_{ab} = 0\,, \tag{1.57}$$

for $\Phi \in \mathrm{End}(\wedge^2\mathfrak{g})$ the orthogonal projection onto the kernel of $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$, and

$$R_{ab}(t^{(0)}_\bullet) = \frac{1}{12} f_a{}^{ce} f_b{}^{df} f_{ef}{}^{g}\{t^{(0)}_c, t^{(0)}_d, t^{(0)}_g\}\,. \tag{1.58}$$

The BRST variation of a generalised line operator involves a term

$$\Theta_{ab,V}\int_\gamma \partial_z c^a \partial_z A^b_\Sigma\,. \tag{1.59}$$

The coefficient $\Theta_{ab,V}$ receives many contributions, two of which are illustrated below.

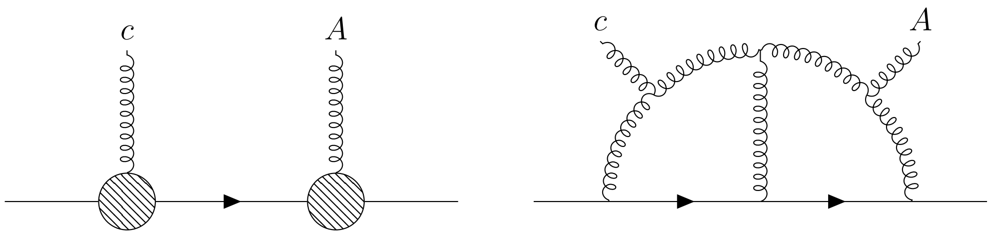


Fig. 1.8 Contributions to the BRST variation of a Wilson line at order $\hbar^2$

Here the shaded discs on the Wilson line indicate a coupling to a level 1 generator $t^{(1)}_{a,V}$ and the label c indicates the insertion of an external ghost field. These two diagrams, together with related ones obtained by permuting the order of vertices on the Wilson line, are the only contributions to (1.59) which either cannot be completely removed by the addition of counterterms or do not necessarily vanish to eliminate other terms in the BRST variation. Together they evaluate to

$$\Big([t^{(1)}_{a,V}, t^{(1)}_{b,V}] - \hbar^2 R_{ab}(t^{(0)}_{\bullet,V})\Big)\int_\gamma \partial_z c^a \partial_z A^b_\Sigma\,. \tag{1.60}$$

That the Lie theoretic part of $R_{ab}(t^{(0)})$ matches that of the second diagram in figure 1.8 is clear. The numerical factor follows from a rather involved calculation.

The appearance of the projection Φ in (1.57) is because any contribution to the BRST variation of the line operator which factors through the commutator $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ can be eliminated by a counterterm. By projecting onto its kernel we extract the part of the BRST variation which cannot be removed in this way. BRST invariance therefore implies the vanishing of

$$\Big([t^{(1)}_{c,V}, t^{(1)}_{d,V}] - \hbar^2 R_{cd}(t^{(0)}_{\bullet,V})\Big)\Phi^{cd}{}_{ab} = 0\,. \tag{1.61}$$

This demonstrates explicitly that line operators in CS_4 are labelled by representations of the Yangian, $\mathcal{Y}(\mathfrak{g})$.

Framing anomaly

In the bulk, line operators which curve in Σ suffer from an anomaly, coming from the Feynman diagram

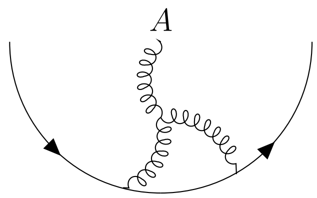


Fig. 1.9 Framing anomaly

This diagram is singular when all three vertices coincide, and so requires regulating. The regularised contribution, modulo redefinitions of the classical Wilson line, is not gauge invariant. Fortunately this anomaly can be cancelled by continuously shifting the spectral parameter as the Wilson line curves, so that

$$z - \frac{\hbar \mathbf{h}^\vee}{2\pi}\varphi \tag{1.62}$$

is constant. Here $\mathbf{h}^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and φ is the angle between the Wilson line and some framing of Σ, increasing as the Wilson line rotates clockwise.

For this to be consistent it is essential that Σ admits a framing, i.e., a global non-vanishing vector field. This excludes $\Sigma = S^2$ for example, but $\Sigma = \mathbb{R}^2$, $S^1 \times \mathbb{R}$ and T^2 are permitted. These latter three examples are those most relevant for the study of quantum integrable spin chains.

One subtle but important consequence of this is that the VEV of the diagram appearing in figure 1.4 does not actually give a solution of the YBE. To understand why, note that, as depicted, the crossing Wilson lines are at a non-vanishing angle to one another. By the framing anomaly, changing this angle is equivalent to shifting the relative values of the spectral parameters at which the Wilson lines are supported, which of course changes the R-matrix. The correlator therefore depends on this angle. If one considers the diagrammatic representation of the YBE in figure 1.5 it's clear that the angle at each crossing is different. Therefore a different matrix is generated at each of the crossing points.

To obtain a true solution to the YBE we should instead compute the VEV of the following configuration of Wilson lines.

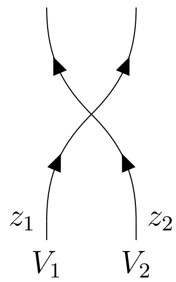


Fig. 1.10 Corrected R-matrix in CS_4

Since the incoming and outgoing Wilson lines are parallel, it's possible to concatenate multiple copies of this configuration without bending any Wilson lines.

If the angle between the Wilson lines in figure 1.4 at the point of crossing is θ, then the correction from the framing anomaly indicates that it will actually compute

$$R_{V_1 \otimes V_2}(z_1 - z_2 + \hbar \mathbf{h}^\vee \theta / 2\pi)\,. \tag{1.63}$$

At first order in $\hbar$ this shift is irrelevant, since there is no z-dependence in the classical limit of the R-matrix. The calculation of the classical r-matrix in section 1.3.3 is therefore still reliable.

For simplicity we will not indicate the necessary curving of Wilson lines in most of our diagrams. It is understood that whenever we compute the VEV of a configuration where line operators tend to infinity that they eventually curve to the vertical. We read diagrams from bottom to top, and horizontal line operators are read from left to right.

RTT presentation

The RTT presentation is an alternative realisation of the Yangian, $\mathcal{Y}(\mathfrak{g})$, constructed using a particular solution to the YBE [71, 68, 35]. Here we review how it is obtained from CS_4.

The idea is to use the rational R-matrix associated to two copies the fundamental representation U of $\mathfrak{g}$ to constrain a generic line operator. Unfortunately the fundamental representation of $\mathfrak{e}_8$ does not lift to the corresponding Yangian, and so we must exclude this case.[5] Consider a generalised line operator

$$\mathcal{W}_V[\gamma, A] = \exp\left(\int_\gamma \sum_{m=0}^{\infty} \frac{1}{m!} t^{(m)}_{a,V} \partial_z^m A^a_\Sigma\right) \tag{1.64}$$

as defined in equation (1.28). For the moment we make no assumptions about the algebra of the $t^{(m)}_{a,V}$. We can probe this generalised line operator with a Wilson line in the representation U. In particular, fixing a basis $\{\langle i|\}_{i=1}^n$ of U and writing $\{|j\rangle\}_{j=1}^n$ for the corresponding dual basis of $U^\vee$, consider the VEV of the following configuration of line operators.

$${T^i}_{j,V}(z) \quad = $$

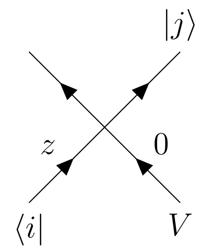


Fig. 1.11 Transfer matrix

[5]By the fundamental we mean the smallest dimension fundamental representation, which for $\mathfrak{e}_8$ happens to be the adjoint. It is interesting to note that $\mathcal{Y}(\mathfrak{e}_8)$ does admit an RTT presentation, which would be interesting to obtain from CS_4 [67].

Here the labels at either end of the Wilson line going from left to right indicate insertions of the states $\langle i|$ and $|j\rangle$. This correlator defines the matrix elements $T^i{}_{j,V}(z)$ of the transfer matrix $T_V(z)$ for $i, j = 1, \ldots, n$. Note that these take values in $\operatorname{End} V$. For the remainder of this section, as long as there is no ambiguity, we suppress the V index on the transfer matrix and its elements.

Expanding $T(z)$ as a power series in $1/z$ allows us to extract the coefficient of $1/z^{m+1}$, which we denote by $\tilde{t}^i{}_j[m]$.

$$T^i{}_j(z) = \delta^i{}_j + \hbar \sum_{m=0}^{\infty} \frac{\tilde{t}^i{}_j[m]}{z^{m+1}} \,. \tag{1.65}$$

We can view these coefficients as defining the generalised line operator $\mathcal{W}_V[\gamma, A]$, instead of the parameters $t_a^{(m)}$ appearing in (1.64). A very slight modification of the calculation in subsection 1.3.3 shows that

$$\tilde{t}^i{}_j[m] = c^{ab}\langle i|t_{a,U}|j\rangle t_b^{(m)} + \mathcal{O}(\hbar)\,, \tag{1.66}$$

with the higher order terms on the right hand side being determined by the Feynman diagram expansion.

We wish to determine the algebra of the coefficients $\tilde{t}^i{}_j[m]$. This is achieved by constraining the transfer matrix using identities relating correlators in CS_4. The most important condition follows from the YBE

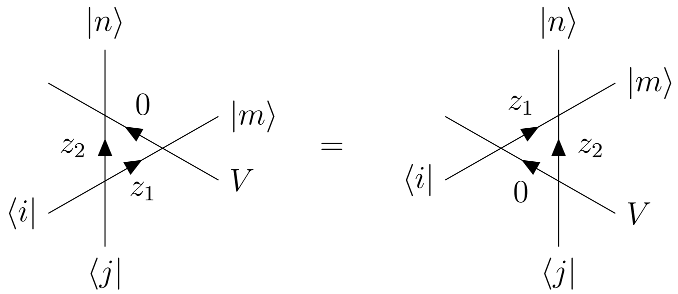


Fig. 1.12 RTT relations

where the lines supported at $z_1, z_2 \in \mathbb{C}$ are in the representation U. Algebraically, this imposes

$$R^{ij}{}_{k\ell}(z_1 - z_2) T^k{}_m(z_1) T^\ell{}_n(z_2) = T^j{}_\ell(z_2) T^i{}_k(z_1) R^{k\ell}{}_{mn}(z_1 - z_2)\,, \tag{1.67}$$

or in short

$$R_{12}(z_1 - z_2)T_1(z_1)T_2(z_2) = T_2(z_2)T_1(z_1)R_{12}(z_1 - z_2)\,. \tag{1.68}$$

The subscripts 1 and 2 indicate on which factor of U the operators act. These are known as the RTT relations.

Expanding these out in terms of the coefficients $\tilde{t}^i{}_j[m]$ introduces relations between them. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ these are sufficient to ensure that the $\tilde{t}^i{}_j[m]$ furnish a representation of $\mathcal{Y}(\mathfrak{gl}_n(\mathbb{C}))$ [68]. It is important to note that the algebra obeyed by these coefficients is not identical to that satisfied by the $t_a^{(m)}$. Indeed, the isomorphism between these two presentations is given by the Feynman diagram expansion (1.66).

In order for the $\tilde{t}^i{}_j[m]$ to determine a representation of $\mathcal{Y}(\mathfrak{g})$ for $\mathfrak{g}$ simple the RTT relation (1.68) must be supplemented by further constraints. In the case $\mathfrak{g} = \mathfrak{sl}_n$ the extra condition required is [101, 68]

$$\varepsilon_{i_1 i_2 \dots i_n} T^{i_1}{}_1(z) T^{i_2}{}_2(z + \hbar) \cdots T^{i_n}{}_n(z + (n-1)\hbar) = \mathbb{1}\,, \tag{1.69}$$

for $\varepsilon_{i_1 i_2 \dots i_n}$ the $\mathfrak{sl}_n$-invariant totally antisymmetric tensor appearing in $U^{\otimes n}$. The object on the left is known as the quantum determinant, $\mathrm{qdet}(T)(z)$ of the transfer matrix. Together with the RTT relations (1.68), the condition $\mathrm{qdet}(T) = \mathbb{1}$ ensures that the $\tilde{t}^i{}_j[m]$ furnish a representation of the Yangian $\mathcal{Y}(\mathfrak{sl}_n(\mathbb{C}))$ [68]. Equation (1.69) has the effect of removing the centre of the algebra defined solely by the RTT relations [126].

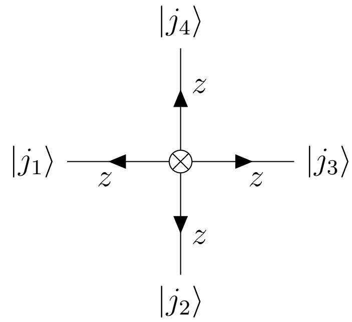


Fig. 1.13 Example vertex in CS_4

In CS_4, the quantum determinant arises [51] from a vertex which joins together n Wilson lines each in the defining representation U of $\mathfrak{sl}_n(\mathbb{C})$. They are connected using the totally antisymmetric tensor $\varepsilon \in U^{\otimes n}$. Classically, to meet at the vertex each of these Wilson lines must be supported at the same value $z \in \mathbb{C}$. The relevant configuration is illustrated in figure 1.13 for the case $n = 4$. Algebraically the vertex is given by

$$\varepsilon_{i_1 i_2 \dots i_n} \mathcal{W}_U[\gamma_1, A]^{i_1}{}_{j_1} \mathcal{W}_U[\gamma_2, A]^{i_2}{}_{j_2} \cdots \mathcal{W}_U[\gamma_n, A]^{i_n}{}_{j_n} . \tag{1.70}$$

In the quantum theory it suffers from an anomaly analogous to the framing anomaly for curved Wilson lines. This can be made to vanish by fixing the angles between the accumulating Wilson lines to be equal. Our diagram above reflects this condition.

Using this vertex, the constraint on the quantum determinant (1.69) can also be represented pictorially. For example, in the case $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ it follows from the equivalence of the following two configurations of Wilson lines

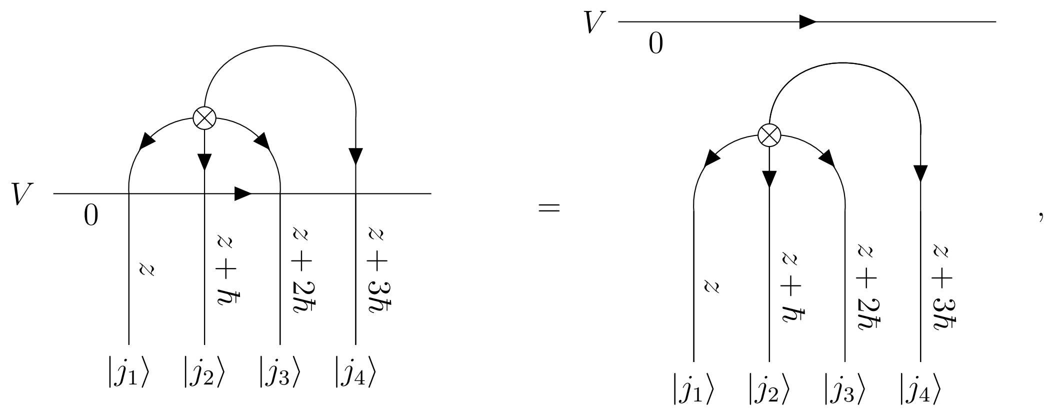


Fig. 1.14 Quantum determinant

where the spectral parameter of the vertex is $z + \hbar$ on both sides. Note the important role of the framing anomaly in supplying the different arguments of the transfer matrices in the quantum determinant (1.69), a consequence of the fact that all the U lines are parallel as they head out to infinity, and so have to curve so as to make equal angles when they meet at the vertex. This provides an alternative demonstration that Wilson lines in CS_4 for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ are labelled by representations of the Yangian.

More generally, whenever the tensor product of some representations V_i of $\mathfrak{g}$ contains a copy of the trivial representation, Wilson lines in these representations can accumulate at a vertex which extracts the invariant part of $\bigotimes V_i$. In this way, gauge invariant networks can be built up from combinations of vertices and Wilson lines. In the quantum theory, such vertices will suffer from anomalies. These can be made to vanish either by fixing the angles between the incoming Wilson lines, or by changing their spectral parameters at order $\hbar$. For totally antisymmetric or symmetric invariants, such as the example above for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, it's enough to ensure that the angles between the incoming Wilson lines are the same.

Using vertices built from the representation U it's possible to obtain RTT presentations of the Yangian for the remaining simple Lie algebras, with the exception of $\mathfrak{e}_8$. For example, $U^{\otimes 2}$ contains a copy of the trivial representation for $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$. This is simply the inverse of the defining symmetric bilinear. The anomaly at the corresponding vertex therefore vanishes if we take the incoming Wilson lines to be opposite to one another. Manipulations analogous to those appearing in figure 1.14 then impose a constraint on the transfer matrix, giving the RTT presentation of $\mathcal{Y}(\mathfrak{so}_n(\mathbb{C}))$.

Remark. The transfer matrix is really nothing more than an R-matrix with one representation fixed to be U. In the same way that vertices built from invariant tensors in U constrain the transfer matrix, a generic R-matrix $R_{V_1\otimes V_2}(z)$ must obey identities arising from vertices involving either V_1 or V_2. The simplest example of this occurs if we take $V_1 = V_2 = U$ for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Then the R-matrix obeys the quantum determinant constraint in both factors.

For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $U = V_1 = V_2$ the constraints on the quantum determinant together with the YBE are enough to uniquely determine the quasi-classical R-matrix given the classical r-matrix [51]. This result extends in a natural way to more general quasi-classical R-matrices. Constraints coming from vertices lift the redundancy in solutions to the YBE under $R(z) \sim F(z,\hbar)R(z)$. (Recall that $F(z) = 1 + \mathcal{O}(\hbar)$ a formal power series in $\hbar$ with coefficients in the ring of meromorphic functions on $\mathbb{C}$.)

The coproduct also has a natural interpretation in the RTT presentation. We have seen that it arises from the OPE of two bulk line operators. Probing this with a Wilson line in the representation U, we have the equivalence of the following two diagrams

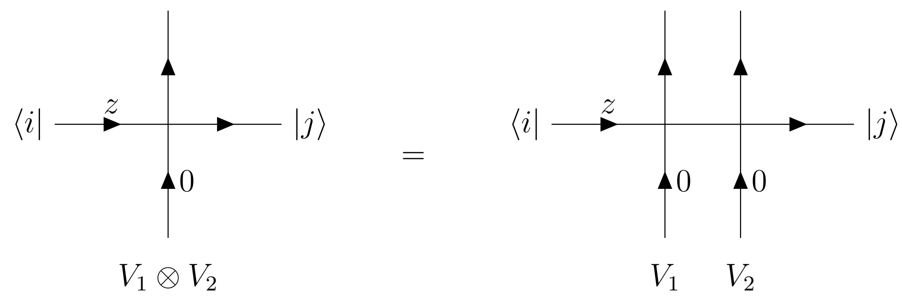


Fig. 1.15 OPE in the RTT presentation

leading to the following identity

$$T^i{}_{j,V_1\otimes V_2} = T^i{}_{k,V_1}(z) \otimes T^k{}_{j,V_2}(z)\,, \tag{1.71}$$

where we've made the representations explicit. More abstractly

$$\Delta_\hbar(T^i{}_j(z)) = T^i{}_k(z) \otimes T^k{}_j(z)\,. \tag{1.72}$$

(Here we are not merely suppressing the representation index, but are viewing $T^i{}_j(z)$ as taking values in $\mathcal{Y}(\mathfrak{g})$.)

Finally, we comment on a condition involving the transfer matrix referred to as unitarity. Consider the configurations illustrated below.

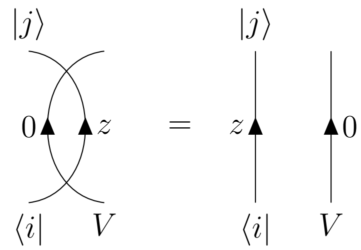


Fig. 1.16 Unitarity

In the usual way these are manifestly equivalent in CS_4. This is diagram is somewhat less straightforward to interpret algebraically, since we do not yet know how to express the VEV of the following configuration in terms of the transfer matrix

$$U^i{}_j(z) \qquad =$$

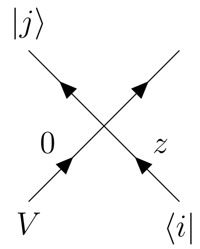


Fig. 1.17 The inverse of the transfer matrix

In fact unitarity is telling us how it should be interpreted. It obeys

$$T^i{}_k(z)U^k{}_j(z) = \delta^i{}_j\,, \tag{1.73}$$

and so it is natural to identify $U(z) = T^{-1}(z)$. Using the framing anomaly it is also possible to rewrite $U(z)$ in terms of the transfer matrix for the representation dual to V. The above configuration will appear in section 2.3.4 when we discuss the RTT presentation for twisted Yangians.

Remark. Whilst unitarity does not constrain the transfer matrix, it does impose the relation

$$R_{12}(z)R_{21}(-z) = \mathbb{1}_{V_1\otimes V_2} \tag{1.74}$$

on R-matrices. This is analogous to the type II Reidemeister move in knot theory.

1.3.5 Trigonometric and elliptic cases

We now turn our attention to the cases $(C,\omega) = (\mathbb{C}/2\pi i\mathbb{Z}, \mathrm{d}z)$ and $(\mathrm{T}^2_\tau, \mathrm{d}z)$. We review how these generate trigonometric and elliptic solutions to the YBE respectively, as demonstrated in [50, 51].

Trigonometric case

Let $(C,\omega) = (\mathbb{C}/2\pi i\mathbb{Z}, \mathrm{d}z)$. As a smooth manifold C is diffeomorphic to a cylinder $S^1 \times \mathbb{R}$. It's sometimes convenient to use coordinates $u = \exp z$ in terms of which $(C,\omega) = (\mathbb{C}^*, \mathrm{d}u/u)$. We can view $\mathbb{C}^*$ as $\mathbb{CP}^1$ with the points $u = 0, \infty$ removed in the standard stereographic coordinates. Our considerations continue to be local in Σ and so we may as well take $\Sigma = \mathbb{R}^2$. (It's still necessary that Σ admits a framing.)

To study CS_4 on $\Sigma \times C$ we must first identify suitable boundary conditions on A as $\operatorname{Re} z \to \pm\infty$. These should be chosen so that varying the action on the support of the classical equations of motion generates no boundary terms. In terms of u we have

$$\begin{aligned}\delta S[A] &= \frac{i}{4\pi}\int_{\mathcal{V}} \frac{\mathrm{d}u}{u} \wedge \mathrm{d}\mathrm{tr}(\delta A \wedge A) + \frac{i}{2\pi}\int_{\mathcal{V}} \frac{\mathrm{d}u}{u} \wedge \mathrm{tr}(\delta A \wedge F(A)) \\ &= \frac{1}{2}\int_\Sigma \mathrm{tr}(\delta A_\Sigma \wedge A_\Sigma|_{u=\infty} - \delta A_\Sigma \wedge A_\Sigma|_{u=0})\,,\end{aligned} \tag{1.75}$$

where we discarded the term involving $F(A)$, and we have made use of the identity

$$\bar{\partial}_C\left(\frac{\mathrm{d}u}{u}\right) = 2\pi i\left(\delta^2(u)\,\mathrm{d}^2u - \delta^2\left(\frac{1}{u}\right)\frac{\mathrm{d}^2u}{|u|^4}\right). \tag{1.76}$$

(The terms on the right hand side are really just 2-form δ-functions supported at $u = 0, \infty$ in $\mathbb{CP}^1$ respectively.) Certainly requiring $A = 0$ at $u = 0, \infty$ ensures that both boundary terms vanish, but this is too stringent: the combination

$$\mathrm{tr}(\delta A_\Sigma \wedge A_\Sigma) \tag{1.77}$$

then vanishes to at least second order at $u = 0, \infty$, but it's sufficient for it to vanish to first order.

Instead we demand that A_Σ takes values in isotropic subspaces $\mathfrak{g}_\pm \subset \mathfrak{g}$ with respect to tr as $\operatorname{Re} z \to \pm\infty$. Compatibility with gauge transformations necessitates that $\mathfrak{g}_\pm$ be Lie subalgebras of $\mathfrak{g}$. Infinitesimal gauge transformations are then similarly restricted to lie in these subalgebras as $\operatorname{Re} z \to \pm\infty$. In the interest of imposing the least restrictive condition possible we choose $\mathfrak{g}_\pm$ to be Lagrangian. Finally, to ensure the classical equations of motion do not admit deformations at $A = 0$, allowing us to perform perturbation theory around this vacuum, we must have $\mathfrak{g}_- \cap \mathfrak{g}_+ = \emptyset$. This implies that $\mathfrak{g} = \mathfrak{g}_- \dotplus \mathfrak{g}_+$, where the symbol $\dotplus$ denotes the direct sum as vector spaces. A triple $(\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+)$ obeying all of these conditions is known as a Manin triple. If the invariant bilinear on $\mathfrak{g}$ is non-degenerate it provides an identification $\mathfrak{g}_- \cong \mathfrak{g}_+^\vee$.

We can use this identification to extend a basis $\{t_A\}_{A=1}^{\dim \mathfrak{g}_+}$ of $\mathfrak{g}_+$ to a basis of all of $\mathfrak{g}$ by writing $\{\bar{t}^A\}_{A=1}^{\dim \mathfrak{g}_+}$ for the dual basis viewed as a basis of $\mathfrak{g}_-$. We assume summation convention on repeated indices. The inverse of the bilinear tr can then be expressed as

$$c = t_A \otimes \bar{t}^A + \bar{t}^A \otimes t_A = c_{+,-} + c_{-,+} \,, \tag{1.78}$$

where $c_{\pm,\mp} \in \mathfrak{g}_\pm \otimes \mathfrak{g}_\mp$.

Assuming these boundary conditions for a given Manin triple $(\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+)$, CS_4 generates a trigonometric solution to the YBE as the VEV of crossing Wilson lines. The associated classical r-matrix appearing at order $\hbar$ is

$$r(z) = \frac{1}{e^z - 1} c_{-,+} + \frac{e^z}{e^z - 1} c_{+,-} \,. \tag{1.79}$$

It obeys the CYBE for $\mathfrak{g}$. This was obtained in [50] by studying the formal properties of the order $\hbar$ contribution to the VEV of crossing Wilson lines, rather than by performing an explicit computation involving the propagator. Indeed, we believe that the propagator of CS_4 with these boundary conditions in the Lorenz type gauge introduced in section 1.3.3 remains unknown.

The problem of generating trigonometric solutions to the BYBE therefore reduces to identifying Manin triples. Unfortunately a generic simple Lie algebra cannot be given the structure of a Manin triple. (Note, for example, that Manin triples are always even dimensional.) It is, however, always possible to construct a Manin triple from any given simple Lie algebra $\mathfrak{g}_0$ by forming the direct sum of Lie algebras

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \widetilde{\mathfrak{h}} \,, \tag{1.80}$$

where $\widetilde{\mathfrak{h}}$ is a second copy of the Cartan subalgebra of $\mathfrak{g}_0$. Note that the Lie algebra $\mathfrak{g}$ is no longer simple since it has a non-trivial centre $\widetilde{\mathfrak{h}}$. The $\mathfrak{g}$-invariant bilinear used in defining the action of CS_4 is

$$\mathrm{tr}_{\mathfrak{g}} = \mathrm{tr}_{\mathfrak{g}_0} + \mathrm{tr}_{\mathfrak{g}_0}|_{\widetilde{\mathfrak{h}}}\,, \tag{1.81}$$

where $\mathrm{tr}_{\mathfrak{g}_0}$ denotes the minimal $\mathfrak{g}_0$-invariant bilinear on $\mathfrak{g}_0$, and $\mathrm{tr}_{\mathfrak{g}_0}|_{\widetilde{\mathfrak{h}}}$ denotes its restriction to the Cartan interpreted as a bilinear on $\widetilde{\mathfrak{h}}$. To determine suitable Lagrangian subalgebras, we pick a decomposition $\mathfrak{g}_0 = \mathfrak{n}_- \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+$, where $\mathfrak{h}$ is a choice of Cartan and $\mathfrak{n}_\pm$ are the subalgebras of positive and negative root spaces for a given base. (In chapter 2 we use $\mathfrak{h}$ to denote the fixed point subalgebra of an involutive automorphism. When there is any risk of ambiguity we instead denote the latter by $\mathfrak{p}$.) We choose

$$\mathfrak{g}_+ = \mathfrak{n}_+ \dotplus \Big\{(H, i\widetilde{H}) \,|\, H \in \mathfrak{h}\Big\}\ . \tag{1.82}$$

Here we are implicitly fixing an identification $\mathfrak{h} \cong \widetilde{\mathfrak{h}}$ between the Cartan $\mathfrak{h}$ of $\mathfrak{g}_0$ and the centre $\widetilde{\mathfrak{h}}$ of $\mathfrak{g}$. The other Lagrangian subalgebra $\mathfrak{g}_-$ is then chosen to be

$$\mathfrak{g}_- = \mathfrak{n}_- \dotplus \Big\{(H, iM(H)) \,|\, H \in \mathfrak{h}\Big\}\,, \tag{1.83}$$

where $M : \mathfrak{h} \to \widetilde{\mathfrak{h}}$ allows for a different identification $\mathfrak{h} \cong \widetilde{\mathfrak{h}}$. It is natural to view M as an element of $\mathrm{Aut}\,\widetilde{\mathfrak{h}}$ which compares the two identifications, and we will regularly abuse notation by interpreting M in this way. For $\mathfrak{g}_-$ to be isotropic M must be chosen to be orthogonal, and for $\mathfrak{g}_+$ and $\mathfrak{g}_-$ to be disjoint it must not have $+1$ as an eigenvalue. An archetypal choice is $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$. M uniquely determines a bivector $A \in \wedge^2\widetilde{\mathfrak{h}}$ by

$$Ac^{-1} = \Big(M + \mathbb{1}_{\widetilde{\mathfrak{h}}}\Big)\Big(M - \mathbb{1}_{\widetilde{\mathfrak{h}}}\Big)^{-1}\,. \tag{1.84}$$

Here we abuse notation by writing c for its projection onto $\mathfrak{h} \otimes \mathfrak{h}$, and are using it to raise and lower indices in the Cartan. The above map is known as the Cayley transform, and its image consists of all bivectors such that $Ac^{-1} \in \mathrm{End}\,\widetilde{\mathfrak{h}}$ does not have a ± 1 eigenvalue.

In order to find an explicit expression for the r-matrix we must introduce a basis of $\mathfrak{g}$ adapted to this Manin triple. Let $\{e_\mu, f_\mu\}_{\mu\in\Phi_+} \cup \{h_\mu\}_{\mu\in\Delta}$ be the standard Chevalley basis of $\mathfrak{g}_0$ with respect to the base Δ. Here Φ_+ denotes the set of positive roots. By adjoining $\{\widetilde{h}_\mu\}_{\mu\in\Delta}$ we can extend this to a basis of $\mathfrak{g}$.

The classical r-matrix given in equation (1.79) specialises to

$$\begin{aligned} r(z) = \frac{1}{e^z - 1} \sum_{\alpha\in\Phi_+} n_\alpha^{-1} f_\alpha \otimes e_\alpha + \frac{e^z}{e^z-1} \sum_{\alpha\in\Phi_+} n_\alpha^{-1} e_\alpha \otimes f_\alpha \\ + \frac{1}{2}\frac{e^z+1}{e^z-1} \sum_{\mu,\nu\in\Delta} c^{\mu\nu}(h_\mu \otimes h_\nu + \widetilde{h}_\nu \otimes \widetilde{h}_\mu) + \frac{1}{2} \sum_{\mu,\nu\in\Delta} A^{\mu\nu}(h_\mu \otimes h_\nu - \widetilde{h}_\mu \otimes \widetilde{h}_\nu) \\ - \frac{i}{2} \sum_{\mu,\nu\in\Delta} (c^{\mu\nu} - A^{\mu\nu}) h_\mu \otimes \widetilde{h}_\nu + \frac{i}{2} \sum_{\mu,\nu\in\Delta} (c^{\mu\nu} + A^{\mu\nu}) \widetilde{h}_\mu \otimes h_\nu \,, \end{aligned} \tag{1.85}$$

for $n_\alpha = \mathrm{tr}(f_\alpha e_\alpha)$. To obtain a solution to the CYBE for $\mathfrak{g}_0$ we must choose both Wilson lines to be in the same 1 dimensional representation of $\widetilde{\mathfrak{h}}$.

Example 1.3.4. Let $\mathfrak{g}_0 = \mathfrak{sl}_n(\mathbb{C})$, and consider the R-matrix generated as the VEV of crossing Wilson lines in the fundamental representation U and uncharged under the $\widetilde{\mathfrak{h}}$. We also set $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$. Writing $\{E_i{}^j\}_{i,j=1}^n$ for the standard basis of U, The corresponding classical r-matrix is

$$\begin{aligned} r_{U\otimes U}(z) = \frac{1}{e^z-1} \sum_{i>j} E_i{}^j \otimes E_j{}^i + \frac{e^z}{e^z-1} \sum_{i<j} E_i{}^j \otimes E_j{}^i \\ + \frac{1}{2}\frac{e^z+1}{e^z-1} \left(\sum_i E_i{}^i \otimes E_i{}^i - \frac{1}{n} \sum_{i,j} E_i{}^i \otimes E_j{}^j \right). \end{aligned} \tag{1.86}$$

This is the classical r-matrix for Jimbo's trigonometric R-matrix [93]

$$\begin{aligned} R(z) \sim \frac{e^z - e^{-2\hbar}}{e^z-1} \sum_i E_i{}^i \otimes E_i{}^i + e^{-\hbar} \sum_{i\neq j} E_i{}^i \otimes E_j{}^j \\ + \frac{1-e^{-2\hbar}}{e^z-1} \sum_{i>j} E_i{}^j \otimes E_j{}^i + \frac{(1-e^{-2\hbar})e^z}{e^z-1} \sum_{i<j} E_i{}^j \otimes E_j{}^i \,, \end{aligned} \tag{1.87}$$

originally discovered in [38, 14]. For $n = 2$ this is the R-matrix describing the XXZ spin chain and 6 vertex model [106]. Switching on charges for $\widetilde{\mathfrak{h}} \cong \mathbb{C}$ corresponds to turning on electric fields in the latter [174, 155].

Strictly speaking in [51] it's only argued that there's a bijection between trigonometric solutions to the YBE equation and the data used in defining CS_4. To see that $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$ leads to the standard Jimbo trigonometric R-matrix observe that in this case the theory has an enhanced symmetry. The Lagrangian is invariant under the composition of a reflection in Σ with the map $z \mapsto -z$ in C, appropriately lifted to the adjoint bundle. In the trigonometric case this lift must respect the boundary conditions by swapping $\mathfrak{g}_\pm$. For $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$ a suitable choice is to act on the fibres with the involutive automorphism

$$\sigma : \mathfrak{g}_0 \dotplus \widetilde{\mathfrak{h}} \to \mathfrak{g}_0 \dotplus \widetilde{\mathfrak{h}}\,, \qquad (X, \widetilde{H}) \mapsto (-X^{\mathrm{t}}, \widetilde{H})\,. \tag{1.88}$$

This symmetry implies that

$$R_{12}(z) = (R_{21}(z))^{\mathrm{t}_1\mathrm{t}_2}\,. \tag{1.89}$$

Adapting the proof of uniqueness in [51], we find that there is a unique R-matrix compatible with this constraint up to a possible non-linear reparametrisation of the parameter $\hbar$. This can be understood as renormalizing $\hbar$, and by tuning its bare value we can ensure that (1.87) holds. This tuning will not affect the results of chapter 3 since we only work to first order in $\hbar$. Varying M, or equivalently the bivector A, has the affect of 'twisting' Jimbo's trigonometric R-matrix in the sense of [149]. This includes the possibility of generalising M to a formal power series in $\hbar$.

Remark. It was observed in [60] that the boundary conditions discussed above are not in fact that most general which preserve diffeomorphism invariance in the Σ direction. Reconsidering equation (1.75), we can see that it's sufficient for the boundary term at $u = 0$ cancel the one at $u = \infty$. It would be interesting to see what solutions of the YBE are generated for the more general boundary conditions they propose.

Elliptic case

We now turn our attention to $(C, \omega) = (\mathrm{T}^2_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \mathrm{d}z)$ for τ the modular parameter. In this case CS_4 generates elliptic solutions to the YBE.

Since T^2_τ is compact and $\mathrm{d}z$ is free from poles, varying the action does not generate boundary terms supported at points in C. There is therefore no need to introduce boundary conditions as were required in the trigonometric case.

We do, however, run into different issues. Firstly, there now exist topologically non-trivial G-bundles over $\mathcal{V}$. Since we study CS_4 perturbatively there is no need to sum over all topologies in the path integral. Nonetheless, we must choose a particular topological G-bundle on which the partial connection A is defined. Our considerations are local in Σ, and so we take this bundle to be the pullback of a topological G-bundle over T^2_τ which we denote by P. Such bundles are determined by an element of $\pi_1(G)$.

Secondly, we must fix an isolated background partially holomorphic structure d' around which to do perturbation theory. Here by isolated we mean that the tangent space to the moduli space of partially holomorphic G-bundles at the vacuum must be trivial. For simplicity we take this background partially holomorphic structure to be the pullback of a holomorphic structure on P.[6] Therefore to define CS_4 we must fix P an isolated holomorphic G-bundle over T^2_τ. For P to be isolated it must have a discrete automorphism group.

Unfortunately the obvious choice of a trivial holomorphic G-bundle $P \cong \mathrm{T}^2_\tau \times G$ does not work, since it is not isolated. In fact the only viable candidates exist for $G = \mathrm{PSL}_n(\mathbb{C})$ with $n \geq 2$. The topology of the G-bundle is then determined by an element of $\zeta \in \mathbb{Z}_n \cong \pi_1(G)$. If we choose ζ to be a generator of $\mathbb{Z}_n$ then there is a unique stable holomorphic structure on P, and it is indeed isolated [50].

This holomorphic structure on P is described by the $(0,1)$ component of a flat connection, which is determined by a map $\pi_1(\mathrm{T}^2_\tau) \cong \mathbb{Z}^2 \to \mathrm{PSL}_n(\mathbb{C})$ up to conjugacy. This in turn is determined by a conjugacy class in the set of pairs of commuting elements in $\mathrm{PSL}_n(\mathbb{C})$. Let

$$A^\alpha{}_\beta = \epsilon^\alpha \delta_{\alpha,\beta}\,, \qquad B^\alpha{}_\beta = \delta_{\alpha,\beta+1} \tag{1.90}$$

be matrices in $\mathrm{SL}_n(\mathbb{C})$ for $\epsilon = \exp(2\pi i/n)$ an n^{th} root of unity. These obey

$$ABA^{-1}B^{-1} = \epsilon \tag{1.91}$$

Viewed as elements of $\mathrm{PSL}_n(\mathbb{C})$, A and B therefore commute. We can define a flat connection on P with holonomy A^ζ and B around the two cycles. Forgetting the $\partial_C + A_C$ component of this connection we get a holomorphic G-bundle over T^2_τ with automorphism group $\mathbb{Z}_n^2$. This is the vacuum we expand around.

We denote the vector bundle associated to P using the adjoint action of G on $\mathfrak{g}$ by $\mathrm{Ad}\,P$. It will be useful to have a more explicit description of this bundle and its sections. Pulling back $\mathrm{Ad}\,P$ by $q : \mathbb{C} \to \mathrm{T}^2_\tau \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ gives the trivial bundle $q^*\mathrm{Ad}\,P \cong \mathbb{C} \times \mathfrak{sl}_n(\mathbb{C})$. Note that this is an isomorphism of holomorphic vector bundles. We can then realise $\mathrm{Ad}\,P$ as the quotient of $\mathbb{C} \times \mathfrak{sl}_n(\mathbb{C})$ by the equivalence relation $(z, X) \sim (z + a + b\tau, \mathrm{conj}(A^{-\zeta a}B^{-b})(X))$ for $a, b \in \mathbb{Z}$. Sections of $\mathrm{Ad}\,P$ can therefore be represented by maps $s : \mathbb{C} \to \mathfrak{sl}_n(\mathbb{C})$ with quasi-periodicity

$$s(z + a + b\tau) = \mathrm{conj}(A^{\zeta a}B^b)(s(z)) \tag{1.92}$$

[6] If $\pi_1(\Sigma)$ is non-vanishing then we could also include the possibility of non-trivial holonomy around the cycles. This allows CS_4 to generate spin chains with twisted period boundary conditions. We do not consider this possibility here.

for $a, b \in \mathbb{Z}$. To understand the holomorphic structure of this bundle better it's useful to introduce a basis of $\mathfrak{sl}_n(\mathbb{C})$ which diagonalises the action of $\operatorname{conj} A^\zeta, \operatorname{conj} B$. The essentially unique choice is $\{t_{i,j}\}_{(i,j)\in\mathcal{I}_n}$ for $t_{i,j} = B^{\zeta^{-1}i}A^{-j}$ and $\mathcal{I}_n = \mathbb{Z}_n^2 \setminus \{(0,0)\}$. In particular

$$\operatorname{conj}(A^\zeta)(t_{i,j}) = \epsilon^i t_{i,j}\,, \qquad \operatorname{conj}(B)(t_{i,j}) = \epsilon^j t_{i,j}\,. \tag{1.93}$$

The structure constants in this basis are given by

$$[t_{i,j}, t_{k,l}] = \left(\epsilon^{-\zeta^{-1}ij} - \epsilon^{-\zeta^{-1}kl}\right)[t_{i+j}, t_{k+l}]\,. \tag{1.94}$$

Let us decompose a section of $\operatorname{Ad} P$, interpreted as a map $s : \mathbb{C} \to \mathfrak{sl}_n(\mathbb{C})$, into its components with respect to this basis:

$$s = \sum_{(i,j)\in\mathcal{I}_n} s^{i,j} t_{i,j}\,. \tag{1.95}$$

Then the condition on its quasi-periodicity implies that

$$s^{i,j}(z + a + b\tau) = \epsilon^{ia+bj} s^{i,j}(z) \tag{1.96}$$

for $a, b \in \mathbb{Z}$. Therefore $s^{i,j}$ determines a section of a holomorphic line bundle $\mathcal{L}_{i,j}$ over T^2_τ. $\mathcal{L}_{i,j}$ has vanishing Chern class and complex structure corresponding to the point $(i + j\tau)/n$ in the Jacobian variety. This is compatible with the Lie algebra structure on the fibres of $\operatorname{Ad} P$ since we have the obvious isomorphism $\mathcal{L}_{(i,j)} \otimes \mathcal{L}_{(k,\ell)} \cong \mathcal{L}_{(i+k,j+\ell)}$.

Therefore as a holomorphic bundle

$$\operatorname{Ad} P \cong \bigoplus_{(i,j)\in\mathcal{I}_n} \mathcal{L}_{(i,j)}\,. \tag{1.97}$$

From now on we write its sections as $s = \sum_{(i,j)\in\mathcal{I}_n} s^{i,j} t_{i,j}$ where the $s^{i,j} : \mathbb{C} \to \mathbb{C}$ have quasi-periodicities (1.96).

Having fixed the vacuum and hence the topological G-bundle on which the gauge field defines a partial connection, CS_4 generates an elliptic R-matrix as the VEV of crossing Wilson lines. The corresponding elliptic r-matrix appearing at order $\hbar$ is

$$r(z) = \frac{1}{n} \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-\zeta^{-1}ij} t_{i,j} \otimes t_{-i,-j} w_{i,j}(z)\,. \tag{1.98}$$

Here the $w_{i,j}(z)$ are meromorphic functions on $\mathbb{C}$ characterised by the fact that they have a simple pole with unit residue at $z = 0$ and quasi-periodicities

$$w_{i,j}(z+1) = \epsilon^i w_{i,j}(z)\,, \qquad w_{i,j}(z+\tau) = \epsilon^j w_{i,j}(z)\,. \tag{1.99}$$

Explicitly

$$w_{i,j}(z) = \sum_{p,q\in\mathbb{Z}_n^2} \frac{\epsilon^{ip+jq}}{z-p-q\tau}\,. \tag{1.100}$$

Formally $r(z_1 - z_2)$ is a meromorphic section of $\mathrm{Ad}\,P \boxtimes \mathrm{Ad}\,P = \pi_1^*\mathrm{Ad}\,P \otimes \pi_2^*\mathrm{Ad}\,P$ over $C \times C$.

The expression (1.98) for the classical r-matrix in the particular case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ was obtained in [50] by arguing from the formal properties it must satisfy, rather than by an explicit computation involving the propagator. Indeed, we do not believe the propagator of CS_4 has been obtained in the Lorenz type gauge discussed in section 1.3.3. It is straightforward to generalise this argument to obtain equation (1.98) which holds for any $n \geq 2$.

Example 1.3.5. In [51] it's proved that there's a unique elliptic R-matrix with associated classical r-matrix (1.98) in the tensor product of two copies of the fundamental of $\mathfrak{sl}_n(\mathbb{C})$. For $\zeta = 1$ we infer the following result which must hold to all orders in $\hbar$:

$$R(z) \sim \sum_{i,j\in\mathbb{Z}_n} \epsilon^{-ij} \frac{\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](z+\hbar/n|\tau)\,\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](\hbar/n|\tau)}{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z|\tau)\,\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](\hbar/n|\tau)} T_{i,j} \otimes T_{-i,-j}\,. \tag{1.101}$$

(Strictly this is true only up to a non-linear reparametrisation of $\hbar$, but by tuning its bare value we can ensure that the above holds. This tuning does no affect the results of chapter 3.) Here we have introduced the basis $\{T_{i,j}\}_{i,j\in\mathbb{Z}_n}$ of $\mathrm{End}(\mathbb{C}^n)$ where $T_{i,j}$ with $(i,j) \not\equiv (0,0)$ is the image of $t_{i,j}$ in the representation U and $T_{0,0} = \mathbb{1}_U$. We have also made use of the θ-functions

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) = \sum_{m\in\mathbb{Z}} \exp(\pi i(m+a)^2\tau + 2\pi i(m+a)(z+b))\,. \tag{1.102}$$

The object appearing on the right hand side of (1.101) is Belavin's $\mathbb{Z}_n$ symmetric R-matrix [20]. For $n = 2$ it describes the XYZ spin chain and 8-vertex model [18].

1.3.6 CS$_4$ for CIFT

We now briefly review the CS$_4$ description of 2d CIFTs originally introduced in [53], concentrating on the case of the PCM. (For an introduction to 2d CIFT see, e.g., [13, 157].)

By inserting defects in CS$_4$ and then eliminating the partial connection A using its classical equations of motion one can obtain a wide range of 2d CIFTs. Those relevant to the current work are associated to 'disorder defects', i.e., modifications to the boundary conditions on A to allow poles. For CS$_4$ to remain gauge invariant such poles can only be tolerated at zeros of ω. We will only be concerned with the case $C = \mathbb{CP}^1$, so that

$$\omega = \varphi(z)\,\mathrm{d}z \tag{1.103}$$

for some meromorphic function $\varphi(z)$. Note that $\mathrm{d}z$ has a hidden double pole at $z = \infty$. For the resulting CIFT not to be topological it is essential that diffeomorphism invariance in the Σ direction is broken by the boundary conditions.

To illustrate this approach we turn to a simplest example involving two disorder defects in CS$_4$:

Example 1.3.6. Let

$$\omega = \frac{K(z-z_1)(z-z_2)\,\mathrm{d}z}{z^2}\,. \tag{1.104}$$

This has simple zeros at z_1, z_2, and second order poles at $0, \infty$. At the poles we impose the standard boundary condition that A should vanish to first order, and similarly assume gauge transformations are trivial. Fixing a complex structure on Σ we allow a simple pole in $A_\Sigma^{1,0}$ at $z = z_1$ and in $A_\Sigma^{0,1}$ at $z = z_2$. When varying the action there is at most a first order pole in

$$\mathrm{tr}(\delta A_\Sigma \wedge A_\Sigma) \tag{1.105}$$

at both $z = z_1, z_2$. The resulting δ-functions are compensated by the zeros in ω.

The choice of a complex structure partially breaks topological invariance in the Σ direction. To see how, let us first introduce holomorphic coordinates $w, \bar{w}$ on Σ. Then consider performing an infinitesimal diffeomorphism in the Σ direction along

$$\xi = \xi^w \partial_w + \xi^{\bar{w}} \partial_{\bar{w}}\,. \tag{1.106}$$

We find that

$$\delta A_w = (\mathcal{L}_\xi A)_w = \xi^w \partial_w A_w + \xi^{\bar{w}} \partial_{\bar{w}} A_w + A_w \partial_w \xi^w + A_{\bar{w}} \partial_w \xi^{\bar{w}}\,. \tag{1.107}$$

For this to be consistent with the requirement that A_w be free from poles at $z = z_2$ it is essential that $\partial_w \xi^{\bar{w}}$ vanishes to first order there. Similarly $\partial_{\bar{w}} \xi^w$ must vanish to first order at $z = z_1$. Therefore, the theory is invariant under holomorphic and antiholomorphic maps, generated by $\xi^w(w)\partial_w$ and $\xi^{\bar{w}}(\bar{w})\partial_{\bar{w}}$ respectively, which are constant in $C = \mathbb{CP}^1$. These descend to complexified conformal symmetries of the resulting 2d CIFT.

To determine this we first fix the gauge by writing

$$\bar{A}_C = \hat{\sigma}^{-1}\bar{\partial}_C\hat{\sigma} \tag{1.108}$$

for $\hat{\sigma} : \mathcal{V} \to G$. (This is always possible under a stability assumption on the G-bundle.) Under a finite gauge transformation with parameter g, which is required to obey $g|_{z=0} = g|_{z=\infty} = \text{id.}$, we find that $\hat{\sigma} \mapsto \hat{\sigma} g^{-1}$. This demonstrates that $\hat{\sigma}$ is essentially arbitrary with the exception of its values at $z = 0, \infty$. There is a further redundancy in $\hat{\sigma}$ under $\hat{\sigma} \mapsto h\hat{\sigma}$ for $h : \Sigma \to G$, although this is an ambiguity in our parametrisation of the gauge equivalence classes of $\bar{A}_C$ rather than a gauge symmetry of CS_4. It can be used to set $\hat{\sigma}|_{z=0} = \text{id.}$. This leaves $\sigma = \hat{\sigma}|_{z=\infty}$ as the gauge invariant data which can be extracted from $\bar{A}_C$. It can be interpreted as the holomorphic Wilson line from z_1 to z_2 [120, 3, 4]. Away from $z = 0, \infty$ we can use the gauge symmetry to fix $\hat{\sigma}$ to take a convenient form. We won't specify it explicitly, but do assume that it is independent of z in neighbourhoods of $z = 0, \infty$ respectively and also invariant under the U(1) symmetry acting by $z \mapsto e^{i\theta} z$.

The next step is to solve the classical equation of motions involving $\bar{A}_C$. To do so we introduce A'_Σ, the partial connection in a forbidden gauge in which $\bar{A}_C = 0$. The parameter of the gauge transformation implementing this gauge is precisely $\hat{\sigma}$, and it is illegal because in general $\hat{\sigma}|_{z=\infty} = \sigma \neq \text{id.}$. Explicitly

$$A = \hat{\sigma}^{-1}(\text{d}' + A'_\Sigma)\hat{\sigma}\,. \tag{1.109}$$

In this gauge the classical equation of motion involving $\bar{A}_C$ simplifies to

$$\bar{\partial}_C A'_\Sigma = 0\,. \tag{1.110}$$

A'_Σ is therefore meromorphic with the same pole structure as A_Σ, and the boundary conditions on A fix

$$A'_\Sigma = \frac{z}{z - z_1} J^{1,0} + \frac{z}{z - z_2} J^{0,1} \tag{1.111}$$

where $J = -\,\mathrm{d}_\Sigma\sigma\sigma^{-1}$. We recognise this as the Lax connection of the PCM. At this point the z-dependence of A is completely determined by equations (1.109) & (1.111), and substituting this back into the action of CS_4 we can perform the integral over z to get an effective 2d theory.

To do so first observe that

$$\mathrm{CS}(X+Y) = \mathrm{CS}(X) + 2\mathrm{tr}(F(X)\wedge Y) - \mathrm{d}\mathrm{tr}(X\wedge Y) + 2\mathrm{tr}(X\wedge Y^2) + \mathrm{CS}(Y)\,. \quad (1.112)$$

Let $\hat{J} = -\,\mathrm{d}'\hat{\sigma}\hat{\sigma}^{-1}$, and set $X = \hat{\sigma}^{-1}\,\mathrm{d}'\hat{\sigma} = -\hat{\sigma}\hat{J}\hat{\sigma}^{-1}$, $Y = \hat{\sigma}A'_\Sigma\hat{\sigma}^{-1}$. Since $Y^3 = 0$,

$$\omega\wedge\mathrm{CS}(Y) = \omega\wedge\mathrm{tr}(\hat{\sigma}^{-1}A'_\Sigma\hat{\sigma}\wedge\mathrm{d}'(\hat{\sigma}^{-1}A'_\Sigma\hat{\sigma}))\,. \quad (1.113)$$

d' must act as $\bar{\partial}_C$, since the two copies of A'_Σ saturate the Σ directions. Furthermore, from equation (1.110) A'_Σ is meromorphic, and so $\bar{\partial}_C A'_\Sigma$ consists of δ-functions supported at $z = z_1, z_2$. These do not contribute the effective action because they are compensated by the zeros in ω. Therefore, modulo terms that do not contribute,

$$\mathrm{tr}(\hat{\sigma}^{-1}A'_\Sigma\hat{\sigma}\wedge\mathrm{d}'(\hat{\sigma}^{-1}A'_\Sigma\hat{\sigma})) = -2\mathrm{tr}(\mathrm{d}'\hat{\sigma}\hat{\sigma}^{-1}\wedge A'_\Sigma\wedge A'_\Sigma) = 2\mathrm{tr}(\hat{J}\wedge A'_\Sigma\wedge A'_\Sigma)\,. \quad (1.114)$$

This exactly compensates the third term in (1.112). Noting that

$$\omega\wedge F(X) = 0\,, \quad (1.115)$$

we conclude

$$\omega\wedge\mathrm{CS}(A) = \frac{1}{3}\omega\wedge\mathrm{tr}(\hat{J}^3) + \omega\wedge\mathrm{d}\mathrm{tr}(\hat{J}\wedge A'_\Sigma)\,. \quad (1.116)$$

Let's concentrate on the second of these terms. Integrating by parts we can move the exterior derivative to ω. We pick up boundary terms from its poles at $z = 0, \infty$, but $\hat{\sigma}|_{z=0} = \mathrm{id.}$ and so only the term at $z = \infty$ contributes. It can be expressed as

$$\frac{iK}{4\pi}\oint_{S^1_{z=\infty}}\frac{(z-z_2)\,\mathrm{d}z}{z}\int_\Sigma\mathrm{tr}(J\wedge J^{1,0}) + \frac{iK}{4\pi}\oint_{S^1_{z=\infty}}\frac{(z-z_1)\,\mathrm{d}z}{z}\int_\Sigma\mathrm{tr}(J\wedge J^{0,1}) \quad (1.117)$$

for $S^1_{z=\infty}$ a small circle around $z = \infty$ in C. This readily evaluates to

$$\frac{K(z_2-z_1)}{2}\int_\Sigma\mathrm{tr}(J^{1,0}\wedge J^{0,1})\,. \quad (1.118)$$

Now consider the first term in (1.116). It contributes

$$\frac{K}{12\pi}\int_{\mathcal{V}}\frac{(z-z_1)(z-z_2)\,\mathrm{d}z}{z^2}\wedge\mathrm{tr}(\hat{J}^3)\,. \tag{1.119}$$

Although we haven't specified $\hat{\sigma}$ exactly, we have assumed that it's invariant under the U(1) action on $C=\mathbb{CP}^1$ fixing $z=0,\infty$. This allows us to perform the integral over the phase of z directly to obtain

$$-\frac{K(z_1+z_2)}{6}\int_{\Sigma\times[0,1]}\mathrm{tr}(\tilde{J}^3)\,. \tag{1.120}$$

Here $\tilde{J}=-\tilde{\mathrm{d}}\tilde{\sigma}\tilde{\sigma}^{-1}$ for $\tilde{\sigma}$ a smooth homotopy from id. to σ and $\tilde{\mathrm{d}}$ the exterior derivative on $\Sigma\times[0,1]$. Putting this together with (1.117) gives

$$\frac{iK(z_1-z_2)}{4}\int_\Sigma\mathrm{tr}(J\wedge *J)-\frac{K(z_1+z_2)}{6}\int_{\Sigma\times[0,1]}\mathrm{tr}(\tilde{J}^3)\,. \tag{1.121}$$

This is the action of the PCM, a prototypical example of an CIFT [70, 145]. Its associated Lax connection is

$$\mathcal{L}=A'_\Sigma=\frac{z}{z-z_1}J^{1,0}+\frac{z}{z-z_2}J^{0,1}\,. \tag{1.122}$$

A number of features of the PCM are manifest from the CS_4 perspective

- Classical conformal invariance of the theory follows straightforwardly from the fact that the boundary conditions depend only on a choice of complex structure on Σ. It is well known that the conformal symmetry is broken in the quantum theory except in the case of WZW_2 [167]. This can be recovered in the $z_1\to 0$ and $z_2\to\infty$ limits with Kz_2 held constant. In the limit $z_2\to z_1$ we are left with just the WZW term, which is consistent with the fact that the boundary conditions become topological.

- The left and right global symmetries of the PCM have a natural interpretation in CS_4. They correspond to 'forbidden' gauge transformations which do not vanish at $z=0,\infty$, but instead take a constant value. Under such a transformation

$$\sigma\mapsto g|_{z=0}^{-1}\hat{\sigma}g|_{z=\infty}\,. \tag{1.123}$$

  In the WZW_2 case these symmetries are significantly enhanced: we can relax the condition that $g|_{z=0}$, $g|_{z=\infty}$ be constant and instead allow them to be holomorphic and antiholomorphic maps $\Sigma \to G$. They correspond to the left and right loop group actions respectively:

$$\sigma \mapsto g|_{z=0}(w)^{-1}\sigma g|_{z=\infty}(\bar{w})\,. \tag{1.124}$$

This example captures the essential features of a general reduction from CS_4 with disorder defects to a 2d CIFT. First one fixes the gauge by writing

$$\bar{A}_C = \hat{\sigma}^{-1}\bar{\partial}_C\hat{\sigma} \tag{1.125}$$

for $\hat{\sigma} : \mathcal{V} \to G$. In all relevant cases $\hat{\sigma}$ can be expressed in terms of a number of group or Lie algebra valued fields on Σ. Then one solves the classical equations of motion involving $\bar{A}_C$ by introducing A'_Σ as in equation (1.109). It implies that A'_Σ is meromorphic in $C = \mathbb{CP}^1$, as seen in equation (1.110). The boundary conditions on A then fix $\bar{A}'_\Sigma$ in terms of the fields appearing in $\hat{\sigma}$. At this point the z-dependence of A is completely determined, and substituting it into the action leads to a 2d CIFT. We identify $\mathcal{L} = A'_\Sigma$ as the Lax connection of the theory.

By varying ω and the corresponding boundary conditions imposed on A at its poles and zeros an abundance of 2d CIFTS have been obtained in this way. Those relevant to the present text are:

- The coupled integrable σ models of [58, 59]. To obtain these choose

$$\omega \propto \frac{\prod_{j=1}^{n}(z-q_j)(z-\tilde{q}_j)}{\prod_{i=1}^{n+1}(z-p_i)^2}\,\mathrm{d}z \tag{1.126}$$

  with the standard boundary condition at the double poles. Fixing a complex structure on Σ, we tolerate simple poles in $A^{1,0}_\Sigma$ and $A^{0,1}_\Sigma$ at $z = q_j$ and $\tilde{q}_j$ respectively.

- The trigonometric deformation of the PCM as introduced in [53]. To obtain this choose

$$\omega \propto \frac{(z-z_1)(z-z_2)}{z(z-z_+)(z-z_-)}\,\mathrm{d}z\,. \tag{1.127}$$

  At the simple poles use the trigonometric boundary conditions introduced in subsection 1.3.5, i.e., we require that

$$A_\Sigma|_{z=0,z_-} \in \mathfrak{g}_-\,, \qquad A_\Sigma|_{z=\infty,z_+} \in \mathfrak{g}_+\,. \tag{1.128}$$

for $\mathfrak{g} = \mathfrak{g}_- \dotplus \mathfrak{g}_+$ a Manin triple. We tolerate simple poles in A_w and $A_{\bar{w}}$ at $z = z_1$ and z_2 respectively. Unfortunately this deformation coincides with the PCM off-shell, but in [60] a slight generalisation led to the bi-Yang-Baxter deformation.

See [53, 60] for many further examples and more information.

Chapter 2

Gauge theory and boundary integrability

Solutions to the YBE are central to the study of quantum integrable models, appearing as the S-matrices of QIFTS and as R-matrices governing the vertex interactions of quantum integrable spin chains.

In his landmark paper [39] Cherednik studied factorised scattering in the presence of a boundary. This led him to a constraint relating the bulk S-matrix of an QIFT to the scattering matrix of excitations off the boundary. It is now known as the boundary Yang-Baxter equation (BYBE). Sklyanin subsequently developed this formalism in [152], and made the connection to quantum integrable spin chains with boundary. Since then the BYBE has been studied extensively.

In this chapter we will largely be concerned with rational solutions to the BYBE. These arise in QIFT as the boundary scattering matrix of the PCM [112] and non-linear Schrödinger model [152, 81]. In the context of quantum integrable spin chains, rational R-matrices underpin the XXX Heisenberg model and its higher rank generalisations. Remarkably, they also play a role in the AdS/CFT correspondence [26, 88, 111] where they describe open strings coupled to giant gravitons.

The first rational solution to the BYBE, solving it for Yang's rational R-matrix, and hence associated to the fundamental of $\mathfrak{sl}_n(\mathbb{C})$, appeared in [39]. This also happens to be the most general solution [125, 7]. Rational solutions to the BYBE for the fundamental representations of the remaining classical Lie algebras were classified in [112, 6].

In much the same way that the Yangian describes rational solutions to the YBE, algebras known as twisted Yangians describe rational solutions to the BYBE. They are associated to involutive automorphisms of the Lie algebra $\mathfrak{g}$, or equivalently symmetric pairs $(\mathfrak{g}, \mathfrak{h})$, and embed as coideal subalgebras of the Yangian of $\mathfrak{g}$. The first twisted Yangians were introduced in [136, 126], where they were defined in the RTT presentation. These are associated to outer automorphisms of $\mathfrak{sl}_n(\mathbb{C})$. The first twisted Yangian corresponding to an inner automorphism appears in [127], also defined in the RTT presentation. Around the same time twisted Yangians were studied in the context of QIFT in [61, 113]. In these works general twisted Yangians are defined as the subalgebras of Yangian charges which remain conserved at the boundary, see also [110] for an introduction. RTT presentations of twisted Yangians for the remaining classical Lie algebras were obtained in [84], though we do not believe they have been described for the exceptional Lie algebras. Intrinsic definitions of twisted Yangians in the J-presentation have been investigated only relatively recently in [24].

In this chapter we will be concerned with extending the gauge theory approach of [42, 50, 51] to described rational solutions of the BYBE. We also demonstrate how the twisted Yangian emerges from this picture. It is largely based on the paper [29].

2.1 Boundary Yang-Baxter equation

We begin by introducing the BYBE which, as the name suggests, is the boundary analogue of the YBE. We assume familiarity with the content of subsection 1.3.1.

Much like the YBE in the bulk, the BYBE plays a double role in the theory of quantum integrable systems with boundary. On the one hand, in 2d QIFT it arises as a constraint on the scattering of particles off the boundary. On the other, its solutions can be used to determine the boundary conditions of an open quantum integrable spin chain. It also plays an important role mathematically, determining coideal subalgebras of infinite dimensional quantum groups. We will be interested in the BYBE in its own right.

We refer to a solution of the BYBE as a K-matrix. This is a linear map $K(z) : V \to V$ for V a complex vector space and z a complex spectral parameter on which it depends meromorphically. The BYBE is the constraint:

$$R_{12}(z_1, z_2)K_1(z_1)R_{21}(z_2, -z_1)K_2(z_2) = K_2(z_2)R_{12}(z_1, -z_2)K_1(z_1)R_{21}(-z_2, -z_1)\,, \tag{2.1}$$

for $R_{12}(z_1, z_2)$ an R-matrix solving the YBE. As usual, the indices just tell us on which factors of the tensor product $V_1 \otimes V_2$ the various operators act. This equation also has a much clearer geometric interpretation, illustrated in figure 2.1. We read this diagram in a similar way to figure 1.3. The only new feature is that a line may reflect off the boundary, here represented by a dotted line. When a line reflects in this way we act with the K-matrix on the corresponding vector space taking the spectral parameter as its argument. Note that when a line reflects of the boundary its spectral parameter changes sign.

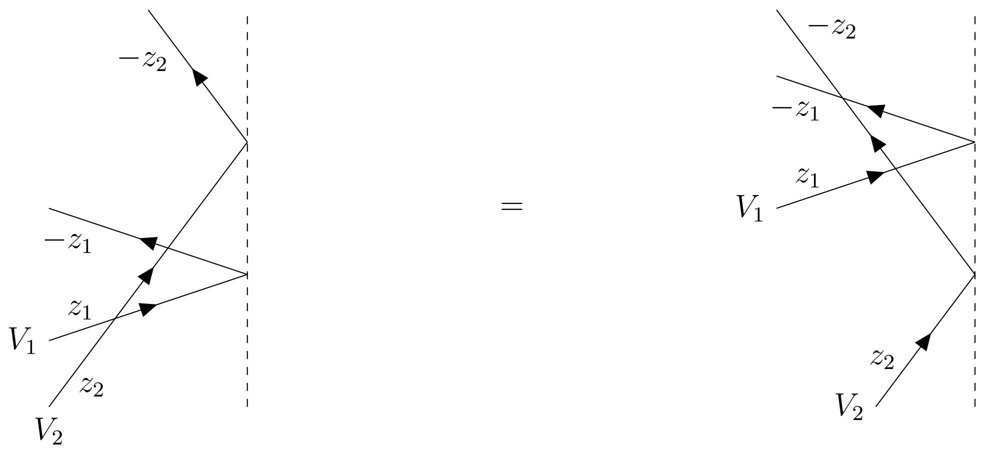


Fig. 2.1 Boundary Yang-Baxter equation

We will find it useful to introduce the notion of a quasi-classical K-matrix solving the BYBE for a quasi-classical R-matrix. This needs to be defined with some care: unlike for quasi-classical R-matrices, the leading $\hbar = 0$ term in a K-matrix is generically not the identity. For example, consider the BYBE for Yang's rational R-matrix. (Recall that this was our motivating example of an R-matrix in subsection 1.3.1. We reviewed in subsection 1.3.3 how it arises in CS_4 as the VEV of two crossing Wilson lines, both in the fundamental of $\mathfrak{sl}_n(\mathbb{C})$.) A family of solutions is given by

$$K(z) = \tau_U + \frac{\Lambda}{z}\mathbb{1}_U \tag{2.2}$$

where $\tau_U^2 \propto \mathbb{1}_U$, and $\Lambda \in \mathbb{C}$ is a free parameter. This was one of the first solutions to the BYBE identified, appearing in [39].

In the rational case we have seen in subsection 1.3.3 that correlators in CS_4 depend on z only through the ratio $\hbar/z$. Even allowing for an overall z-dependent rescaling, if $\tau_U \neq \mathbb{1}_U$ then the above family of K-matrices cannot be obtained via perturbative corrections to the identity. Motivated by this, we take a quasi-classical K-matrix to have the form

$$K(z, \hbar) = \tau_V + \hbar k_V(z) + \mathcal{O}(\hbar^2) \tag{2.3}$$

where $k_V(z)$ is the 'classical k-matrix' and $\tau_V^2 \propto \mathbb{1}_V$.[1] For us to interpret (2.2) as a quasi-classical K-matrix we will need to take $\Lambda = \mathcal{O}(\hbar)$. In fact, we will assume that $\Lambda = \hbar\lambda$ for $\lambda \in \mathbb{C}$ so that the K-matrix is invariant under simultaneous rescalings of z and $\hbar$.

Having introduced τ_V it is natural to require that the R-matrix is compatible with it in an appropriate sense. In particular, we assume that

$$R_{12}(z) = \sigma_1\sigma_2 R_{21}(z) \tag{2.4}$$

for $\sigma_i = \operatorname{conj}\tau_{V_i}$. At first order in $\hbar$ this implies that

$$r_{12}(z) = \sigma_1\sigma_2 r_{21}(z)\,. \tag{2.5}$$

For an abstract classical r-matrix with values in $\mathfrak{g}\otimes\mathfrak{g}$ this condition makes sense if we take τ to be an element of G with the property that $\tau^2 \in Z(G)$, the centre of G. Then $\sigma = \operatorname{conj}\tau$ defines an involutive inner automorphism of $\mathfrak{g}$, and in turn, σ determines a symmetric pair $(\mathfrak{g},\mathfrak{h})$ by $\mathfrak{h} = \{X \in \mathfrak{g} : \sigma(X) = X\}$. (See appendix B.1 for further details.) Representations V of $\mathfrak{g}$ lift to G, and so we can interpret τ_V as the image of τ under such a lift.

Assuming that both the R- and K-matrix are quasi-classical, we can expand the BYBE order-by-order in $\hbar$. The condition appearing at first order holds identically if the classical r-matrix obeys (2.5). A rather complicated constraint appears at order $\hbar^2$, which for an R-matrix obeying (2.4) simplifies so that it depends only on the classical r- and k-matrix. Introducing the 'classical ℓ-matrix' $\ell_V(z) = k_V(z)\tau_V^{-1}$ and applying the identity (2.5), it can be expressed as

$$\begin{aligned}
0 &= [r(z_1-z_2), \sigma_2 r_{12}(z_1+z_2)] \\
&\qquad + [r_{12}(z_1-z_2) - \sigma_2 r_{12}(z_1+z_2), \ell_1(z_1)] \\
&\qquad\qquad + [r_{12}(z_1-z_2) + \sigma_2 r_{12}(z_1+z_2), \ell_2(z_2)] \\
&= [r_{12}(z_1-z_2) + \ell_1(z_1) - \ell_2(z_2), \sigma_2 r_{12}(z_1+z_2) + \ell_1(z_1) + \ell_2(z_2)]\,.
\end{aligned} \tag{2.6}$$

We will refer to this as the classical boundary Yang-Baxter equation (CBYBE). Given that we have been able to understand both σ and the classical r-matrix abstractly, it

[1] We believe this definition first appeared in our paper [29]. It has since be used in [83]. Related concepts have appeared elsewhere. In the rational case quasi-classical K-matrices have implicitly appeared in [127, 84]. In the trigonometric case K-matrices have been constructed using quantum symmetric pairs for q-deformations of quantum affine algebras, see, e.g. [99, 148]. q is a finite complex parameter, but identifying '$q = \exp\hbar$' should lead to quasi-classical K-matrices [35].

is natural to seek a similar interpretation for the classical ℓ-matrix. Whilst, much like the CYBE, the CBYBE involves only the commutator on the spaces $\operatorname{End} V_i$, we will find that it is too restrictive to view $\ell(z)$ as taking values in the Lie algebra $\mathfrak{g}$. Instead, we assume that it maps into the UEA, $U(\mathfrak{g})$. The classical k-matrix is then a right translation of $\ell(z)$ by τ. Fixing a representation V of $\mathfrak{g}$ we recover a classical k-matrix with values in $\operatorname{End} V$.

In seeking solutions to the BYBE we will find it useful to consider a slightly more general class of K-matrices, namely those which describe degrees of freedom living at the boundary. These are linear maps

$$K(z) : V \otimes W \to V \otimes W \tag{2.7}$$

depending meromorphically on the parameter z. They are required to obey a slight generalisation of the BYBE

$$R_{12}(z_1, z_2)K_{13}(z_1)R_{21}(z_2, -z_1)K_{23}(z_2) = K_{23}(z_2)R_{12}(z_1, -z_2)K_{13}(z_1)R_{21}(-z_2, -z_1)\,, \tag{2.8}$$

where the 3 index labels the $\operatorname{End} W$ factor of the tensor product. Diagrammatically we now have

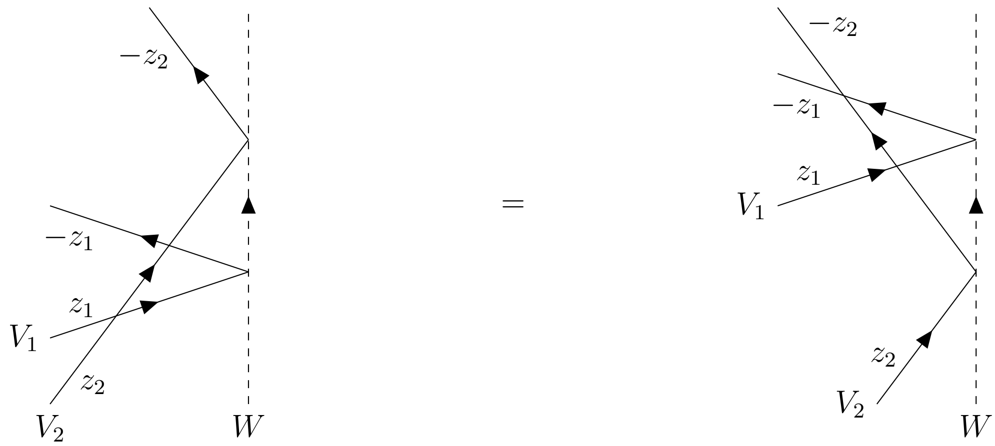


Fig. 2.2 BYBE with boundary degrees of freedom

where the arrow on the dotted line representing the boundary indicates the order in which the K-matrices act on $\operatorname{End} W$.

K-matrices with boundary degrees of freedom are important in studying integrable scattering off impurities that possess internal degrees of freedom, e.g., in the Kondo problem [75, 76]. We recover the case without boundary degrees of freedom if W has dimension 1.

A quasi-classical K-matrix with boundary degrees of freedom is taken to be a formal power series in $\hbar$ with the form

$$K(z) = \tau_V \otimes \mathbb{1}_W + \hbar \ell_{V\otimes W}(z)(\tau_V \otimes \mathbb{1}_W) + \mathcal{O}(\hbar^2) \tag{2.9}$$

where we continue to assume that $\tau_V^2 \propto \mathbb{1}_V$ and refer to $\ell_{V\otimes W}(z)$ as a classical ℓ-matrix. Substituting this into equation (2.8) gives a slightly modification of the CBYBE at order $\hbar^2$:

$$\begin{aligned} 0 &= [r(z_1 - z_2), \sigma_2 r_{12}(z_1 + z_2)] + [\ell_{13}(z_1), \ell_{23}(z_2)] \\ &\quad + [r_{12}(z_1 - z_2) - \sigma_2 r_{12}(z_1 + z_2), \ell_{13}(z_1)] \\ &\qquad + [r_{12}(z_1 - z_2) + \sigma_2 r_{12}(z_1 + z_2), \ell_{23}(z_2)] \\ &= [r_{12}(z_1 - z_2) + \ell_{13}(z_1) - \ell_{23}(z_2), \sigma_2 r_{12}(z_1 + z_2) + \ell_{13}(z_1) + \ell_{23}(z_2)] \\ &\quad - [\ell_{13}(z_1), \ell_{23}(z_2)] \,. \end{aligned} \tag{2.10}$$

This makes sense for an abstract classical ℓ-matrix taking values in $U(\mathfrak{g}) \otimes U(\mathfrak{h})$, where we recall that $\mathfrak{h} = \{X \in \mathfrak{g} : \sigma(X) = X\}$. Fixing representations V and W of $\mathfrak{g}$ and $\mathfrak{h}$ gives a classical ℓ-matrix which maps into $\mathrm{End}(V \otimes W)$.

We refer to quasi-classical K-matrices (both with and without boundary degrees of freedom) as rational, trigonometric and elliptic according to the family of the quasi-classical R-matrix for which they satisfy the BYBE. Similarly, we refer to classical ℓ- and k-matrices according to the family of the classical r-matrix for which they satisfy the CBYBE.

For a given classical ℓ-matrix and representation W of $\mathfrak{h}$ it is not always the case that there exists a corresponding quasi-classical K-matrix. Only certain representations are permissible. For each family of classical ℓ-matrices there is an associated coideal subalgebra of the bulk quantum group. In the rational case these are known as twisted Yangians. (Sometimes these subalgebras are referred to as reflection algebras, with the term twisted Yangian reserved for automorphisms σ which are outer.) Given a quasi-classical R-matrix in the bulk, a corresponding quasi-classical K-matrix will exist only if W lifts to a representation of the appropriate coideal subalgebra.

As in the bulk, the BYBE is homogeneous and so its solutions are defined only up to multiplication by a function $G(z, \hbar)$. (Strictly G is a formal power series in $\hbar$ whose coefficients are meromorphic functions in z.) This is an equivalence relation and we refer its equivalence classes as K-matrices up to scale. A representative in each class can be specified by imposing further constraints on the K-matrix.

2.2 Rational K-matrices from CS_4

In this section we show how rational solutions to the BYBE can be obtained from CS_4 on a $\mathbb{Z}_2$ orbifold. We assume familiarity with the content of subsections 1.3.2 & 1.3.3.

2.2.1 CS_4 on a $\mathbb{Z}_2$ orbifold

We now explore how to extend CS_4 so that it can generate solutions to the BYBE.

Since we wish to describe integrable models with boundary, we replace the 2d smooth manifold Σ with the 2d smooth manifold with boundary $\overline{\Sigma}$. It might seem natural to study CS_4 on the manifold with boundary $\overline{\mathcal{V}} = \overline{\Sigma} \times C$, and we will see shortly that we could take this view, but it will be more illuminating to instead work on a $\mathbb{Z}_2$ orbifold built from $\overline{\Sigma}$ and C.

We start by noting that any manifold with boundary $\overline{M}$ can be viewed as a $\mathbb{Z}_2$ quotient of the manifold M obtained by gluing together a copy of $\overline{M}$ and its mirror image along their common boundary. Formally $M = \overline{M}_{\mathrm{L}} \sqcup \overline{M}_{\mathrm{R}} / \{\partial \overline{M}_{\mathrm{R}} \sim \partial \overline{M}_{\mathrm{L}}\}$ for $\overline{M}_{\mathrm{L}}$ a copy of $\overline{M}$ and $\overline{M}_{\mathrm{R}}$ a copy of $-\overline{M}$ as oriented manifolds. M is sometimes referred to as the double of $\overline{M}$. It admits a natural $\mathbb{Z}_2$ action by the reflection $(p_{\mathrm{L}}, p_{\mathrm{R}}) \mapsto (p_{\mathrm{R}}, p_{\mathrm{L}})$, which fixes the common boundary. We write $\overline{M} \cong M/\mathbb{Z}_2$. Applying this construction to $\overline{\Sigma}$ we obtain its double which we denote by Σ. Let $\mathcal{R}$ be the reflection generating its $\mathbb{Z}_2$ action.

In this and the next chapter we will take $(C, \omega) = (\mathbb{C}, \mathrm{d}z)$, $(\mathbb{C}/2\pi i\mathbb{Z}, \mathrm{d}z)$ or $(\mathrm{T}^2_\tau, \mathrm{d}z)$, which we have seen in section 1.3 lead to rational, trigonometric or elliptic R-matrices.

We then form $\mathcal{V} = \Sigma \times C$ and quotient by the $\mathbb{Z}_2$ action generated by the composition of the reflection $\mathcal{R}$ on Σ with the map $z \mapsto -z$ on C, which we denote by $\mathcal{P}$. This choice of $\mathbb{Z}_2$ action is motivated by the fact that in the pictorial representation of the BYBE, given in figures 2.1 & 2.2, the spectral parameter changes sign when a line reflects off the boundary. We will study CS_4 on the $\mathbb{Z}_2$ orbifold $\widetilde{\mathcal{V}} = \mathcal{V}/\mathbb{Z}_2$. It is important to note that $\widetilde{\mathcal{V}}$ is not itself the double of some manifold with boundary. Indeed, its set of singular points, $L = \partial\overline{\Sigma} \times \{2z = 0\}$, has codimension 3, not 1. In the rational, trigonometric and elliptic cases $2z = 0$ has one, two and four solutions respectively. L therefore consists of a discrete set of lines. These will play an important role in what follows.

CS_4 will continue to be local in Σ, so we may take $\Sigma = \mathbb{R}^2$ and $\overline{\Sigma} = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$. Then concretely $\mathcal{V} = \mathbb{R}^2 \times C$ and

$$\mathcal{P} : v = (x, y, z, \bar{z}) \mapsto \mathcal{P}v = (-x, +y, -z, -\bar{z})\,. \tag{2.11}$$

To place a gauge theory on $\widetilde{\mathcal{V}}$ we must lift the action of $\mathcal{P}$ to the principal G-bundle. Assuming this is trivial, we can simply act on the fibres with an involutive automorphism $\sigma : G \to G$, i.e., a homomorphism from $\mathbb{Z}_2$ to $\operatorname{Aut} G$. This determines an involutive automorphism of $\mathfrak{g}$, which we also denoted by σ. As mentioned in section 2.1, involutive automorphisms of $\mathfrak{g}$ are in bijection with symmetric pairs $(\mathfrak{g}, \mathfrak{h})$. In particular, $\mathfrak{h}$ is the fixed point subalgebra of σ. See appendix B.1 for further details. In this work we will always take σ to be an inner automorphism, so that $\sigma = \operatorname{conj} \tau$ for some $\tau \in G$ with $\tau^2 \in Z(G)$. Non-perturbatively, we would expect to sum over all choices of σ as part of the quantum gauge theory. However, here we work only perturbatively and so it makes sense to fix a choice of σ.

A partial connection on the G-bundle over $\widetilde{\mathcal{V}}$ determined by σ is defined to be a partial connection on $\mathcal{V}$ obeying

$$A = \sigma \mathcal{P}^* A \,. \tag{2.12}$$

In terms of the components of A this is

$$\begin{aligned} A_x(x,y,z,\bar{z}) &= -\sigma\big(A_x(-x,y,-z,-\bar{z})\big)\,, \\ A_y(x,y,z,\bar{z}) &= +\sigma\Big(A_y(-x,y,-z,-\bar{z})\Big)\,, \\ A_{\bar{z}}(x,y,z,\bar{z}) &= -\sigma\Big(A_{\bar{z}}(-x,y,-z,-\bar{z})\Big)\,. \end{aligned} \tag{2.13}$$

Similarly, an infinitesimal gauge transformation on $\widetilde{\mathcal{V}}$ is defined to be an infinitesimal gauge transformation on $\mathcal{V}$ obeying

$$\varepsilon = \sigma \mathcal{P}^* \varepsilon \,. \tag{2.14}$$

This implies that

$$\mathrm{d}_\Sigma \varepsilon + \bar{\partial}_C \varepsilon + [A, \varepsilon] = \sigma \mathcal{P}^* \Big(\mathrm{d}_\Sigma \varepsilon + \bar{\partial}_C \varepsilon + [A, \varepsilon] \Big)\,, \tag{2.15}$$

ensuring compatibility between gauge transformations and (2.12).

Along the orbifold singularity L, the condition (2.12) implies that $\iota_L^* A = \sigma \iota_L^* A$, or in components $A_y|_L = \sigma A_y|_L$. In contrast $A_x|_L = -\sigma A_x|_L$ and $A_{\bar{z}}|_L = -\sigma A_{\bar{z}}|_L$. Thus, the component of the gauge field tangent to L is restricted to lie in $\mathfrak{h}$, whilst the normal components live in $\mathfrak{m}$.

We take the action on the orbifold to be

$$S_{\widetilde{\mathcal{V}}}[A] = \frac{1}{|\mathbb{Z}_2|} S_{\mathcal{V}}[A] = \frac{i}{8\pi} \int_{\mathcal{V}} \omega \wedge \operatorname{tr}\Bigg(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \Bigg)\,. \tag{2.16}$$

For this to be consistent it is essential that $S_{\mathcal{V}}[A]$ respects the $\mathbb{Z}_2$ action. Note that

$$\begin{aligned} S_{\mathcal{V}}[\mathcal{P}^*\sigma A] &= \frac{i}{2\pi}\int_{\mathcal{V}} \omega\wedge \mathrm{CS}(\mathcal{P}^*\sigma A) = \frac{i}{4\pi}\int_{\mathcal{V}} \omega \wedge \mathcal{P}^*\mathrm{CS}(\sigma A) \\ &= -\frac{i}{4\pi}\int_{\mathcal{V}} \mathcal{P}^*\Big(\omega\wedge \mathrm{CS}(A)\Big) = -\frac{i}{4\pi}\int_{\mathcal{P}(\mathcal{V})} \omega\wedge\mathrm{CS}(A) = S_{\mathcal{V}}[A]\,. \end{aligned} \tag{2.17}$$

Here in going from the second to the third line we've used the fact that $\mathcal{P}^*\omega = -\omega$ and σ preserves the bilinear tr. In going from the fourth to the final line we've used the fact that $\mathcal{P}$ is orientation reversing, so that $\mathcal{P}(\mathcal{V}) = -\mathcal{V}$ as oriented manifolds.

In the quantum theory we integrate over all field configurations on $\mathcal{V}$ obeying (2.12), which in practice involves constructing a propagator which is sourced by a $\mathbb{Z}_2$-invariant δ-function.[2] In order for perturbation theory to make sense the vacuum $A = 0$ must be an isolated solution to the equations of motion on $\widetilde{\mathcal{V}}$ modulo gauge transformations. This follows directly from the fact that $A = 0$ is an isolated solution on $\mathcal{V}$.

Given that action (2.16) is proportional to the standard action of CS$_4$ (1.11), it is invariant under all of the same symmetries. (These were discussed in subsection 1.3.2.) However, in the orbifold theory they must all commute with the $\mathbb{Z}_2$ action to preserve the configuration space of fields. In particular, the theory is invariant under diffeomorphisms

$$\xi = \xi^x\partial_x + \xi^y\partial_y \tag{2.18}$$

obeying $\mathcal{P}_*\xi = \xi$. The translation symmetry in C generated by ω^{-1} is broken.

It will often be convenient to have a slightly different perspective on the orbifold theory. We can pick a representative of $\widetilde{\mathcal{V}}$ inside $\mathcal{V}$ by removing a codimension 1 surface π chosen so that $\mathcal{V}\setminus\pi = \mathcal{V}_{\mathrm{L}} \sqcup \mathcal{V}_{\mathrm{R}}$ with $\mathcal{P}(\mathcal{V}_{\mathrm{R}}) = -\mathcal{V}_{\mathrm{L}}$. For example, we can take π to be the plane $x = 0$, and then $x < 0$ in $\mathcal{V}_{\mathrm{L}}$ while $x > 0$ in $\mathcal{V}_{\mathrm{R}}$. Explicitly, we have $\mathcal{V}_{\mathrm{L}} \cong \Sigma_{\mathrm{L}} \times C$, where $\Sigma_{\mathrm{L}} = \{(x, y) \in \mathbb{R}^2 : x < 0\}$. Since $\mathcal{P}(x) = -x$, every point in $\widetilde{\mathcal{V}}$ has at least one representative in $\overline{\mathcal{V}}_{\mathrm{L}} = \mathcal{V}_{\mathrm{L}} \sqcup \pi \cong \overline{\Sigma}_{\mathrm{L}} \times C$. However, $\mathcal{P}$ acts non-trivially on π, identifying $(y, z) \sim (y, -z)$. π is therefore itself an orbifold.

The conditions (2.12) imply that a partial connection on $\widetilde{\mathcal{V}}$ is determined by its restriction to $\overline{\mathcal{V}}_{\mathrm{L}}$. Whilst A is unconstrained in $\mathcal{V}_{\mathrm{L}}$, on π it must obey non-local 'boundary' conditions. The simplest of these is

$$A_i(v)|_\pi = \mathbb{P}_i{}^j\sigma\, A_j(\mathcal{P}(v))|_\pi\,, \tag{2.19}$$

[2]See [89] for a rather different approach to studying CS$_3$ on a $\mathbb{Z}_2$ orbifold by replacing the orbifold singularities with Wilson lines.

which is obtained by evaluating the constraints (2.12) on π. (Here we have introduced the symbol $\mathbb{P}_i{}^j = \mathrm{diag}(-,+,-)$ for later convenience. Recall that the indices $i, j, \ldots$ take values in $\{x, y, \bar{z}\}$.) There are further constraints obtained by taking derivatives of (2.12) in the direction normal to π. Similarly, along π gauge transformations obey non-local 'boundary' conditions. One such condition is

$$\partial_i \varepsilon(v)|_\pi = -\sigma\, \mathbb{P}_i{}^j \partial_j \varepsilon(\mathcal{P}(v))|_\pi \,, \tag{2.20}$$

obtained by evaluating (2.15) on π. This guarantees the compatibility of gauge transformations with (2.19).

We can then decompose the action as

$$S_{\mathcal{V}}[A] = S_{\overline{\mathcal{V}}_{\mathrm{L}}}[A] + S_{\mathcal{V}_{\mathrm{R}}}[A] \,. \tag{2.21}$$

For any gauge field pulled back from $\widetilde{\mathcal{V}}$, the condition (2.12) allows us to write

$$S_{\mathcal{V}_{\mathrm{R}}}[A] = S_{\mathcal{V}_{\mathrm{R}}}[\sigma \mathcal{P}^* A] = \frac{i}{4\pi} \int_{\mathcal{V}_{\mathrm{R}}} \omega \wedge \mathrm{CS}(\sigma \mathcal{P}^* A) = \frac{i}{4\pi} \int_{\mathcal{P}(\mathcal{V}_{\mathrm{R}})} \mathcal{P}^* \Big(\omega \wedge \mathrm{CS}(\sigma A)\Big) \tag{2.22}$$

Using $\mathcal{P}^*\omega = -\omega$, $\mathcal{P}(\mathcal{V}_{\mathrm{R}}) = -\mathcal{V}_{\mathrm{L}}$ as oriented manifolds and the fact that σ preserves the invariant bilinear tr, $S_{\mathcal{V}_{\mathrm{R}}}[A] = S_{\mathcal{V}_{\mathrm{L}}}[A]$ and therefore

$$S_{\widetilde{\mathcal{V}}}[A] = \frac{1}{|\mathbb{Z}_2|} S_{\mathcal{V}}[A] = S_{\overline{\mathcal{V}}_{\mathrm{L}}}[A] = \frac{i}{4\pi} \int_{\overline{\mathcal{V}}_L} \omega \wedge \mathrm{CS}(A) \,. \tag{2.23}$$

The conditions (2.19) ensure that the boundary terms obtained when varying the action vanish. To see this, consider an arbitrary variation δA on $\overline{\mathcal{V}}_{\mathrm{L}}$ that also obeys (2.19). We find

$$\delta S_{\overline{\mathcal{V}}_{\mathrm{L}}}[A] = \frac{i}{2\pi} \int_{\overline{\mathcal{V}}_{\mathrm{L}}} \omega \wedge \mathrm{tr}(\delta A \wedge F) - \frac{i}{4\pi} \int_\pi \omega \wedge \mathrm{tr}(\delta A \wedge A) \,. \tag{2.24}$$

Since $\mathcal{P}$ preserves the orientation of π, the same argument as above shows that the boundary term vanishes:

$$\int_\pi \omega \wedge \mathrm{tr}(\delta A \wedge A) = \int_\pi \mathcal{P}^*\omega \wedge \mathrm{tr}(\sigma \mathcal{P}^* \delta A \wedge \sigma \mathcal{P}^* A) = -\int_\pi \omega \wedge \mathrm{tr}(\delta A \wedge A) \,, \tag{2.25}$$

In summary, we have shown that the action on the orbifold can be expressed in terms of an action for a partial connection A which is unconstrained on $\mathcal{V}_{\mathrm{L}}$, and obeys non-local boundary conditions on $\partial\overline{\mathcal{V}}_{\mathrm{L}} = \pi$. There are two advantages to doing this. The first is that quantum field theory on a manifold with boundary is no doubt more familiar than on an orbifold, and the second is that it makes the connection to the manifold with boundary $\overline{\Sigma}$ more explicit.

One major downside of this perspective is that it appears to partially break diffeomorphism invariance in the Σ direction to the subgroup preserving π. We will need diffeomorphisms which do not lie in this subgroup to derive the Sklyanin determinant constraint in subsection 2.3.4.

We now restrict our attention to the rational case, $(C, \omega) = (\mathbb{C}, \mathrm{d}z)$. The trigonometric and elliptic cases will be dealt with in chapter 3. We saw in subsection 1.3.3 that for this choice of C CS_4 has enhanced symmetries which partially survive in the orbifold theory. Constant gauge transformations must obey (2.14) and so take values in $\mathfrak{h}$. The global G symmetry of the theory is therefore broken to $H = \{g \in G : \sigma(g) = g\}$, where we are abusing notation by continuing to write σ for its lift to G. This reduction of the global symmetry algebra will be responsible for the fact that rational solutions to the BYBE are not in general G-invariant. The exception to this is $\sigma = \mathrm{id.}$, for which full G-invariance is maintained. The symmetry under simultaneous rescalings of $\hbar$ and z is unmodified.

Before continuing to the next section we will find it useful to fix a basis $\{t_a\}_{a=1}^{\dim \mathfrak{g}}$ for the Lie algebra $\mathfrak{g}$ which is adapted to the splitting $\mathfrak{g} = \mathfrak{h} \dotplus \mathfrak{m}$. We choose $\{t_\alpha\}_{\alpha=1}^{\dim \mathfrak{h}}$ to be a basis of $\mathfrak{h}$ and similarly $\{t_\mu\}_{\mu=\dim \mathfrak{h}+1}^{\dim \mathfrak{g}}$ to be a basis of $\mathfrak{m}$. Sometimes we will want to distinguish between the abstract Lie algebra $\mathfrak{h}$ and its embedding in $\mathfrak{g}$, and so we instead write $\{b_\alpha\}_{\alpha=1}^{\dim \mathfrak{h}}$ for its basis. The inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is given by $b_\alpha \mapsto t_\alpha$. From now on we use indices from the beginning of the Greek alphabet $\alpha, \beta, \gamma, \ldots$ to index the basis vectors in $\mathfrak{h}$, and indices from the middle of the Greek alphabet $\mu, \nu, \xi, \ldots$ to index the basis vectors in $\mathfrak{m}$. Note that since $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal, raising and lowering indices with tr and $c = \mathrm{tr}^{-1}$ respects which part of the Greek alphabet they're from. We assume summation convention for repeated indices. Further details are included in appendix B.1.

2.2.2 Orbifold operators

Any operator in CS_4 on $\mathcal{V}$ can be viewed as an operator in the orbifold theory simply by restricting it to partial connections obeying the relations (2.12). In this way we can define Wilson lines and generalised line operators on $\widetilde{\mathcal{V}}$. However, it is not necessarily the case that operators which are distinct on $\mathcal{V}$ define different operators on $\widetilde{\mathcal{V}}$. For example, consider an ordinary Wilson line on $\mathcal{V}$. Using the relations (2.12) we have

$$\mathcal{W}_V[\gamma, A] = \mathcal{W}_V[\gamma, \sigma\mathcal{P}^* A] = \mathcal{W}_V[\mathcal{P}(\gamma), \sigma A] = \mathcal{W}_{V^\sigma}[\mathcal{P}(\gamma), A]\,, \tag{2.26}$$

where V^σ is the representation of $\mathfrak{g}$ defined by composing V with σ, i.e., $\rho_{V^\sigma} = \rho_V \circ \sigma$. We can therefore see that $\mathcal{W}_V[\gamma, A]$ is equivalent to $\mathcal{W}_{V^\sigma}[\mathcal{P}(\gamma), A]$ on $\widetilde{\mathcal{V}}$, which is consistent with the fact that γ and $\mathcal{P}(\gamma)$ are identified. It is useful to have a prescription for specifying operators uniquely.

One suitable choice is to represent all operators on $\overline{\mathcal{V}}_{\mathrm{L}}$ using (2.26). Firstly, if γ already happens to lie purely in $\mathcal{V}_{\mathrm{L}}$, then the line operator $\mathcal{W}_V[\gamma, A]$ is of the required form immediately. Conversely, if $\gamma \subset \mathcal{V}_{\mathrm{R}}$ then we use (2.26) to express it as Wilson line supported on $\mathcal{P}(\gamma) \subset \mathcal{V}_{\mathrm{L}}$, as required. The interesting case is where γ crosses π. Suppose this happens at most once, which will certainly be true for straight lines. Then $\gamma = \gamma_{\mathrm{L}} \sqcup \gamma_{\mathrm{R}}$ where $\gamma_{\mathrm{L}} \subset \overline{\mathcal{V}}_{\mathrm{L}}$ and $\gamma_{\mathrm{R}} \subset \mathcal{V}_{\mathrm{R}}$. We can decompose any such Wilson line as

$$\mathcal{W}_V[\gamma, A] = \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\mathcal{W}_V[\gamma_{\mathrm{R}}, A] = \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\mathcal{W}_{V^\sigma}[\mathcal{P}(\gamma_{\mathrm{R}}), A]\,, \tag{2.27}$$

and since $\mathcal{P}(\gamma_{\mathrm{R}}) \subset \mathcal{V}_{\mathrm{L}}$ we have written our orbifold line operator in terms of a representative on $\overline{\mathcal{V}}_{\mathrm{L}}$. We can view this configuration as being made up of an incident Wilson line $\mathcal{W}_V[\gamma_{\mathrm{L}}, A]$ terminating at some point $(y, z) \in \pi$, and a reflected Wilson line $\mathcal{W}_{V^\sigma}[\mathcal{P}(\gamma_{\mathrm{R}}), A]$ emerging from $(y, -z) \in \pi$. Though separated in C, from the point of view of $\overline{\Sigma}_{\mathrm{L}}$ these operators appear to intersect at a point on the boundary.

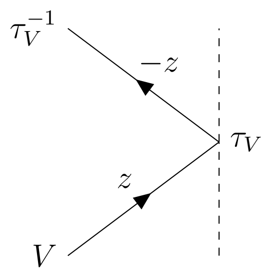


Fig. 2.3 A reflecting Wilson line

Now we exploit the fact that σ is an inner automorphism of $\mathfrak{g}$, i.e. $\sigma = \operatorname{conj}\tau$ for $\tau \in G$ obeying $\tau^2 \in Z(G)$. This allows us to simplify

$$\mathcal{W}_{V^\sigma}[\gamma, A] = \mathcal{W}_V[\gamma, \sigma(A)] = \tau_V \mathcal{W}_V[\gamma, A]\tau_V^{-1}\,, \tag{2.28}$$

which applied to equation (2.27) gives

$$\mathcal{W}_V[\gamma, A] = \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\tau_V \mathcal{W}_V[\mathcal{P}(\gamma_{\mathrm{R}}), A]\tau_V^{-1}\,. \tag{2.29}$$

This only involves Wilson lines in the representation V. From the perspective of $\overline{\Sigma}_{\mathrm{L}}$, we can represent this configuration as in figure 2.3.

From the orbifold perspective it is clear that this operator is classically gauge invariant, but it is also helpful to see this directly on $\overline{\mathcal{V}}_{\mathrm{L}}$. Under an infinitesimal gauge transformation we have

$$\begin{aligned}
&\delta(\mathcal{W}_V[\gamma_{\mathrm{L}}, A]\tau_V \mathcal{W}_V[\mathcal{P}(\gamma_{\mathrm{R}}), A]) \\
&= \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\varepsilon_V(0,y,z)\tau_V \mathcal{W}_V[\mathcal{P}(\gamma_{\mathrm{R}}), A] - \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\tau_V\varepsilon_V(0,y,-z)\mathcal{W}_V[\mathcal{P}(\gamma_{\mathrm{R}}), A] \\
&= \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\varepsilon_V(0,y,z)\tau_V \mathcal{W}_V[\mathcal{P}(\gamma_{\mathrm{R}}), A] - \mathcal{W}_V[\gamma_{\mathrm{L}}, A]\varepsilon_V(0,y,z)\tau_V\mathcal{W}_V[\mathcal{P}(\gamma_{\mathrm{R}}), A] \\
&= 0\,,
\end{aligned} \tag{2.30}$$

as expected. In the second line we've used the condition (2.20) which enforces $\varepsilon(0,y,z) = \sigma(\varepsilon(0,y,-z))$ on π.

Whilst diffeomorphisms in the Σ direction can shift the point of reflection on $\partial\overline{\Sigma}_{\mathrm{L}}$ up and down, they can never be used to pull a single reflecting Wilson line away from the boundary. Note, however, that from the orbifold perspective there's nothing special about our choice of π and it's certainly possible to move Wilson lines through it. In particular, a line operator that curves so as to pass through π an even number of times can be removed from the boundary altogether, simply by choosing a different representative plane π' with $\gamma \cap \pi' = \emptyset$. (We will revisit this idea in subsection 2.3.4.)

It should now be clear how to write general arrangements of Wilson lines on $\widetilde{\mathcal{V}}$ in terms of representatives on $\overline{\mathcal{V}}_{\mathrm{L}}$: whenever any a curve $\gamma \subset \widetilde{\mathcal{V}}$ intersects π transversely, the associated line operator on $\overline{\mathcal{V}}_{\mathrm{L}}$ will reflect off the boundary.

In subsection 1.3.3 we saw that it's natural to introduce a more general class of line operators on $\mathcal{V}$. Classically these are labelled by representations of $\mathfrak{g}[[u]]$, the Lie algebra of formal power series in u with coefficients in $\mathfrak{g}$. An operator of this type is given by

$$\mathcal{W}_V[\gamma, A] = \mathrm{P}\exp\left(\int_\gamma A_{\Sigma,V}^{\mathfrak{g}[[u]]}\right) = \mathrm{P}\exp\left(\int_\gamma \sum_{n=0}^{\infty} \frac{1}{n!} t_{a,V}^{(n)} \partial_z^n A_\Sigma^a\right), \tag{2.31}$$

where $\gamma \subset \mathcal{V}$ is supported at a point in C and V is a representation of $\mathfrak{g}[[u]]$. A calculation similar to (2.26) shows that on the orbifold this line operator is identified with $\mathcal{W}_{V^{\hat{\sigma}}}[\mathcal{P}(\gamma), A]$ where $V^{\hat{\sigma}}$ is the composition of the representation V with the automorphism of $\mathfrak{g}[[u]]$ defined by

$$\hat{\sigma}(t_\alpha^{(n)}) = (-)^n t_\alpha^{(n)}\,, \qquad \hat{\sigma}(t_\mu^{(n)}) = (-)^{n+1} t_\mu^{(n)}\,. \tag{2.32}$$

We can use this to represent any such line operator on $\overline{\mathcal{V}}_{\mathrm{L}}$, just as we have done for ordinary Wilson lines.

We will sometimes refer to the Wilson lines and generalised line operators on $\widetilde{\mathcal{V}}$ introduced above as bulk line operators to distinguish them from those to be defined below.

So far we have seen that map from operators in CS_4 on $\mathcal{V}$ to those in the orbifold theory is not injective. It is not surjective either. This is due to an exceptional class of line operators lying along the orbifold singularity L. In general L is a set of disjoint lines along $\partial\overline{\Sigma}$ supported at the points $\{2z = 0\} \subset C$. In this chapter we are concentrating on the rational case, for which L consists of a single line supported at $z = 0$. Recalling that the pullback of A to L takes values in $\mathfrak{h}$, we can construct Wilson lines along L associated to representations W of $\mathfrak{h}$, rather than $\mathfrak{g}$:

$$\mathcal{W}_W[L, A] = \mathrm{P}\exp\left(\int_L A_{\Sigma,W}\right) = \mathrm{P}\exp\left(\int_L b_{\alpha,W} A_\Sigma^\alpha\right). \tag{2.33}$$

We refer to this as a 'boundary Wilson line' because, from the perspective of $\overline{\Sigma}_{\mathrm{L}}$, it lies entirely within the boundary $\partial\overline{\Sigma}_{\mathrm{L}}$. To illustrate the presence of a line operator on L, we decorate the figure 2.3 with an arrow on the dotted line representing π. We shall see that boundary Wilson lines play an important role in generating K-matrices from CS_4.

Just as we can define generalised bulk line operators which are classically labelled by representations of $\mathfrak{g}[[u]]$, we can also introduce a more general class of line operators along L. To build them we first need to acquaint ourselves with $\mathfrak{g}^\sigma[[u]] = \mathfrak{h}[[u^2]] \oplus u\mathfrak{m}[[u^2]]$, the fixed-point subalgebra of the involution $\hat{\sigma}$ introduced in equation (2.32). It's spanned by $\{b_\alpha^{(2n)}\}_{\alpha=1,n\in\mathbb{Z}_0^+}^{\dim\mathfrak{h}} \cup \{b_\mu^{(2n+1)}\}_{\mu=\dim\mathfrak{h}+1,n\in\mathbb{Z}_0^+}^{\dim\mathfrak{g}}$, with Lie bracket

$$\begin{aligned} [b_\alpha^{(2m)}, b_\alpha^{(2n)}] &= f_{\alpha\beta}{}^\gamma b_\gamma^{(2(m+n))}\,, \\ [b_\alpha^{(2m)}, b_\mu^{(2n+1)}] &= f_{\alpha\mu}{}^\nu b_\nu^{(2(m+n)+1)}\,, \\ [b_\mu^{(2m+1)}, b_\nu^{(2n+1)}] &= f_{\mu\nu}{}^\gamma b_\gamma^{(2(m+n+1))}\,. \end{aligned} \tag{2.34}$$

Explicitly we have the inclusion

$$\mathfrak{g}^\sigma[[u]] \hookrightarrow \mathfrak{g}[[u]]\,, \qquad b_\alpha^{(2n)} \mapsto t_\alpha^{(2n)}\,, \qquad b_\mu^{(2n+1)} \mapsto t_\mu^{(2n+1)}\,. \tag{2.35}$$

It is also possible to give a presentation of $\mathfrak{g}^\sigma[[u]]$ involving only the generators $\{b_\alpha^{(0)}\}_{\alpha=1}^{\dim\mathfrak{h}} \cup \{b_\mu^{(1)}\}_{\mu=\dim\mathfrak{h}+1}^{\dim\mathfrak{g}}$. See [24] for further details. We define representations of $\mathfrak{g}^\sigma[[u]]$ in the same way as for $\mathfrak{g}[[u]]$, i.e., we require that there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$ the images of $b_\bullet^{(m)}$ vanish.

On L the condition $A = \sigma\mathcal{P}^*A$ imposes constraints on the ∂_z derivatives of A. In particular

$$\iota_L^*\partial_z^n A = (-)^n\sigma(\iota_L^*\partial_z^n A) \tag{2.36}$$

which projecting onto $\mathfrak{h}$ and $\mathfrak{m}$ implies

$$\iota_L^*\partial_z^{2n+1}A^{\mathfrak{h}} = 0\,, \qquad \iota_L^*\partial_z^{2n}A^{\mathfrak{m}} = 0\,. \tag{2.37}$$

We can therefore form

$$(\iota_L^*A)^{\mathfrak{g}^\sigma[[u]]} = \sum_{n=0}^\infty \left(\frac{1}{2n!}b_\alpha^{(2n)}\iota_L^*\partial_z^{2n}A^\alpha + \frac{1}{(2n+1)!}b_\mu^{(2n+1)}\iota_L^*\partial_z^{2n+1}A^\mu\right). \tag{2.38}$$

Using this connection on L, we can define a boundary line operator labelled by a representation, W, of $\mathfrak{g}^\sigma[[u]]$.

$$\begin{aligned}\mathcal{W}_W[L,A] &= \mathrm{P}\exp\left(\int_L (\iota_L^*A)_W^{\mathfrak{g}^\sigma[[u]]}\right)\\ &= \mathrm{P}\exp\left(\int_L \sum_{n=0}^\infty\left(\frac{1}{2n!}b_{\alpha,W}^{(2n)}\iota_L^*\partial_z^{2n}A^\alpha + \frac{1}{(2n+1)!}b_{\mu,W}^{(2n+1)}\iota_L^*\partial_z^{2n+1}A^\mu\right)\right).\end{aligned} \tag{2.39}$$

Note that for this to be consistent with invariance under simultaneous rescalings of z and $\hbar$ we must view $b_\bullet^{(m)}$ as being of order $\hbar^m$.

Now let us turn our attention to quantum theory. We have reviewed in section 1.3.3 how the IR freedom of CS$_4$ after gauge fixing provides a local prescription for evaluating correlation functions. This continues to apply in the orbifold theory. Consider a general configuration of Wilson lines on $\widetilde{\mathcal{V}}$, which we choose to represent uniquely on $\overline{\mathcal{V}}_{\mathrm{L}}$. Perhaps by first acting with a diffeomorphism in the Σ direction, we may assume that this configuration consists of reflections and pairwise crossings concatenated together. (There may also be a boundary Wilson line present.) By scaling up the metric in Σ we can see that the only contributions to the VEV come from gluon exchanges in arbitrarily small neighbourhoods of the points in Σ_{L} at which Wilson lines cross and in $\partial\overline{\Sigma}_{\mathrm{L}}$ at which they reflect of the boundary. This leads to a local prescription for evaluating the VEVs of configurations of Wilson lines in the orbifold theory. The basic building blocks are the VEVs of the configuration 1.4 and the one illustrated below. (Note that have included the possibility of a boundary Wilson line.)

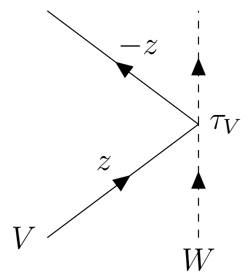


Fig. 2.4 K-matrix from CS_4

The VEV of crossing Wilson lines in Σ_{L} is completely unchanged from the bulk, since by scaling up the metric we can assume that the crossing occurs arbitrarily far away from π. We therefore recover the bulk R-matrix as described in subsection 1.3.3. Note that $A \mapsto \sigma\mathcal{P}^*A$ is a symmetry of CS_4 on $\mathcal{V}$, and so the rational R-matrix obeys

$$R_{12}(z) = \sigma_1\sigma_2 R_{21}(z)\,. \tag{2.40}$$

This is precisely the identity assumed in section 2.1. It holds in the rational case for any inner automorphism σ on account of G-invariance.[3]

Now let us turn our attention to the VEV of the configuration 2.4, which yields a map

$$K(z) : V \otimes W \to V \otimes W\,. \tag{2.41}$$

This is holomorphic away from $z = 0$, a consequence of the formal properties of the path integral. Notice that diffeomorphism invariance in the Σ direction ensures that the VEVs of the two configurations of Wilson lines illustrated in figure 2.5 are the same. It is clear that this implies the BYBE for $K(z)$. In this way CS_4 on $\widetilde{\mathcal{V}}$ generates quasi-classical K-matrices.

Just as quantum corrections to bulk line operators require that they be labelled by representations of the Yangian, so too quantum corrections to boundary line operators in the orbifold theory mean that they are really labelled by representations of the twisted Yangian. In addition, the framing anomaly for curved Wilson lines will necessitate a

[3]This identity will also hold in the trigonometric and elliptic cases, where it is a considerably less trivial constraint.

slight modification of the above realisation of K-matrices in CS$_4$. Both of these issues will be discussed in section 2.3.

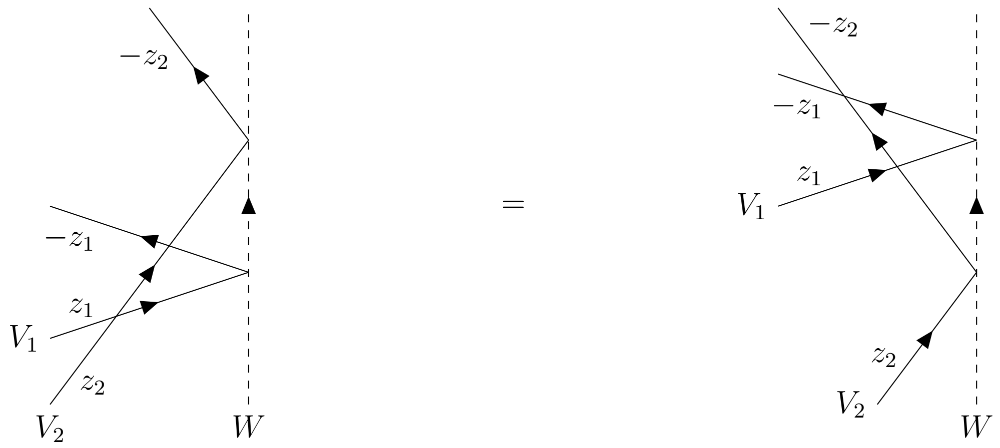


Fig. 2.5 BYBE in CS$_4$

2.2.3 Orbifold propagator

To study quantum CS$_4$ on $\widetilde{\mathcal{V}}$, the first ingredient we need is the propagator $\widetilde{P}$.

In subsection 1.3.3 we computed the propagator of CS$_4$ on $\mathcal{V}$ in the gauge

$$* \, \mathrm{d} * A = \mathrm{D}^i A_i = \partial_x A_x + \partial_y A_y + 4\partial_z A_{\bar{z}} = 0 \,. \tag{2.42}$$

Here $D^i = (\partial_x, \partial_y, 4\partial_{\bar{z}})$ and $*$ is the Hodge star associated to the metric

$$g = \delta_\Sigma + \frac{1}{2}\omega\bar{\omega} = \mathrm{d}x^2 + \mathrm{d}y^2 + \frac{1}{2}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{2.43}$$

on $\mathcal{V}$. We fix the same gauge on the orbifold $\widetilde{\mathcal{V}}$.

The components of the propagator relevant to us are

$$\hbar \widetilde{P}_{ij}(v_1, v_2) = \langle A_i(v_1) \otimes A_j(v_2) \rangle_0 \,, \qquad \hbar \widetilde{Q}_i(v_1, v_2) = \hbar \langle h(v_1) \otimes A_i(v_2) \rangle_0 \,, \tag{2.44}$$

for $h(v)$ the Nakanishi-Lautrup field. ($\widetilde{Q}$ will not play a role in any calculations, but it does appear in the defining equations of $\widetilde{P}$.) They obey

$$\begin{aligned} D^i \widetilde{P}_{ij}(v_1, v_2) &= 0 \,, \\ \varepsilon^{ijk} \partial_j \widetilde{P}_{k\ell}(v_1, v_2) - D^i \widetilde{Q}_\ell(v_1, v_2) &= \frac{\pi}{2}(c\delta^i{}_\ell \delta^4_g(v_1 - v_2) + c^\sigma \mathbb{P}_\ell{}^i \delta^4_g(v_1 - \mathcal{P}v_2)) \,, \\ D^i \widetilde{Q}_i(v_1, v_2) &= -\frac{\pi}{2}(c\delta^i{}_\ell \delta^4_g(v_1 - v_2) + c^\sigma \mathbb{P}_\ell{}^i \delta^4_g(v_1 - \mathcal{P}v_2)) \,. \end{aligned} \tag{2.45}$$

Here the object on the right hand side of the second and third equation is the $\mathbb{Z}_2$-invariant analogue of a δ-function on the orbifold, and $c^\sigma = \sigma_2 c$. The action of $\mathcal{P}$ breaks translation invariance in $\mathcal{V}$, and so we can no longer assume that either $\widetilde{P}$ or $\widetilde{Q}$ depend on their arguments through their difference. $\mathbb{Z}_2$-invariance (2.12) imposes further constraints

$$\begin{aligned} \widetilde{P}_{ij}(v_1, v_2) &= \sigma_1 \mathbb{P}_i{}^k \widetilde{P}_{kj}(\mathcal{P}v_1, v_2)\,, \\ \widetilde{P}_{ij}(v_1, v_2) &= \sigma_2 \mathbb{P}_j{}^\ell \widetilde{P}_{i\ell}(v_1, \mathcal{P}v_2)\,, \end{aligned} \tag{2.46}$$

and similarly for $\widetilde{Q}$. To solve for $\tilde{P}$ recall the bulk propagator

$$P_{ij}(v) = c\Delta_{ij}(v) = -\frac{c}{4\pi}\varepsilon_{ijk} D^k\left(\frac{1}{\|v\|^2}\right). \tag{2.47}$$

Since $A \mapsto \sigma\mathcal{P}^* A$ is a symmetry of the quadratic part of the gauge fixed bulk action, it obeys

$$P_{ij}(v) = \sigma_1\sigma_2 \mathbb{P}_i{}^k \mathbb{P}_j{}^\ell P_{k\ell}(\mathcal{P}v)\,. \tag{2.48}$$

In view of the constraints (2.46), a natural ansatz is

$$\begin{aligned} \widetilde{P}_{ij}(v_1, v_2) &= P_{ij}(v_1 - v_2) + \sigma_2 \mathbb{P}_j{}^\ell P_{i\ell}(v_1 - \mathcal{P}v_2) \\ &= c\Delta_{ij}(v_1 - v_2) + c^\sigma \mathbb{P}_j{}^\ell \Delta_{i\ell}(v_1 - \mathcal{P}v_2)\,. \end{aligned} \tag{2.49}$$

(We make an analogous ansatz for $\widetilde{Q}$.) This manifestly obeys the second constraint in equation (2.46), and the first follows from the identity (2.48). It is straightforward to show that this propagator obeys the defining equation (2.45).

Written out explicitly

$$\widetilde{P}_{ij}(v_1, v_2) = -\frac{c}{4\pi}\varepsilon_{ijk} D^k\left(\frac{1}{\|v_1 - v_2\|^2}\right) - \frac{c^\sigma}{4\pi}\mathbb{P}_j{}^\ell \varepsilon_{i\ell k} D^k\left(\frac{1}{\|v_1 - \mathcal{P}v_2\|^2}\right). \tag{2.50}$$

Remark. If, instead of working equivariantly with respect to the $\mathbb{Z}_2$ action, we choose to work on the manifold with boundary $\overline{\mathcal{V}}_{\mathrm{L}}$ as outlined in section 2.2.1, then we can interpret the orbifold propagator slightly differently. On $\overline{\mathcal{V}}_{\mathrm{L}} \subset \mathcal{V}$ it's a Green's function for the bulk equation of motion (1.36). It also obeys the relevant non-local boundary conditions on π in both arguments which are inherited from those on A appearing in equation (2.19). We can interpret the second term in the propagator as arising from an image charge, where we use the method of images to impose the boundary conditions. Note that the normalizations agree since the factor of $1/|\mathbb{Z}_2|$ in front of the orbifold action (2.16) compensates a factor of 1/2 which appears in the $\mathbb{Z}_2$-invariant δ-function (2.45).

2.2.4 Computing the rational ℓ-matrix

We have seen in subsection 2.2.2 how K-matrices are generated as the VEVs of reflecting Wilson lines in CS_4, perhaps in the presence of a boundary Wilson line. The relevant configuration is illustrated in figure 2.4. Consider the perturbative expansion of such a VEV. The classical limit can be identified by evaluating the operator on the vacuum configuration $A = 0$. Unlike for R-matrices, this does not give $\mathbb{1}_{V\otimes W}$ since we have explicitly inserted τ_V at the boundary, as is required for gauge invariance. We instead find that

$$K(z) = \tau_V \otimes \mathbb{1}_W + \hbar k_{V\otimes W}(z) + \mathcal{O}(\hbar^2)\,. \tag{2.51}$$

This is precisely the form of a quasi-classical K-matrix with boundary degrees of freedom as introduced in section 2.1. We identify $\ell(z) = k(z)(\tau^{-1}\otimes 1)$ as the classical ℓ-matrix.

We now proceed to calculate $k(z)$. We'll start with the simple case where there is no line operator on L, incorporating this later. In the absence of a boundary line operator, the order $\hbar$ contribution comes from the three Feynman diagrams

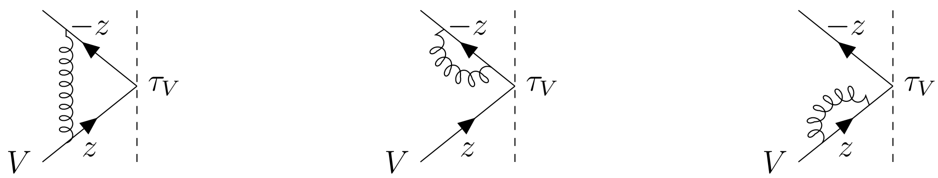


Fig. 2.6 Self-interactions of a reflecting Wilson line

The second two diagrams arise from the interaction of a bulk Wilson line with itself. Such self-interactions of a line operator are usually interpreted as renormalizing it, but here they must be taken to contribute to the K-matrix to ensure that the result is independent of the angle of incidence. We can understand these as arising from the interaction between a bulk Wilson line and its mirror image. They are only significant when a Wilson line gets arbitrarily close to the boundary.

From the discussion in section 2.2.2, we know that this configuration can realised on the orbifold by a straight line crossing the 'boundary' π. Then all three of these contributions can be seen to come from a single diagram, illustrated in figure 2.7. We must remember to include τ_V which acts at the end of the Wilson line.

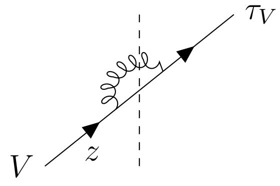


Fig. 2.7 Self-interaction of a bulk Wilson line on the orbifold

The diagram in figure 2.7 contributes

$$t_{a,V} t_{b,V} \tau_V \int_{\infty<s<t<\infty} \mathrm{d}s\,\mathrm{d}t\, \frac{\mathrm{d}\gamma^i}{\mathrm{d}s} \frac{\mathrm{d}\gamma^j}{\mathrm{d}t}\, \widetilde{P}^{ab}_{ij}(\gamma(s),\gamma(t)) \tag{2.52}$$

for $\gamma(s) = (s\cos\theta, s\sin\theta, z, \bar{z})$, where θ is the angle of incidence. Note that a line on the orbifold that is straight with respect to some framing of Σ will appear in $\overline{\Sigma}_{\mathrm{L}}$ to have equal angles of incidence and reflection defined with respect to the induced framing of $\overline{\Sigma}_{\mathrm{L}}$. The three diagrams in figure 2.6 correspond to the integration regions $s<0<t$, $0<s<t$ and $s<t<0$ respectively. Recall the orbifold propagator (2.49)

$$\widetilde{P}_{ij}(v_1, v_2) = c\Delta_{ij}(v_1 - v_2) + c^\sigma \mathbb{P}_j{}^\ell \Delta_{i\ell}(v_1 - \mathcal{P}v_2)\,. \tag{2.53}$$

The first term on the right hand side is simply the bulk propagator, it does not contribute in (2.52) since it is antisymmetric in i,j. We can therefore replace $\tilde{P}$ with the second term, which gives

$$\begin{aligned} &\frac{1}{8\pi} c^{ab} t_{a,V} \tau_V t_{b,V} \sin 2\theta \int_{-\infty<s<t<\infty} \mathrm{d}s\,\mathrm{d}t\, \frac{8\bar{z}}{(\cos^2\theta(t+s)^2 + \sin^2\theta(t-s)^2 + 4|z|^2)^2} \\ &\qquad = \frac{c^{ab} t_{a,V} \tau_V t_{b,V}}{\pi} \int_A \mathrm{d}u\,\mathrm{d}v\, \frac{2\bar{z}}{(u^2+v^2+4|z|^2)^2}\,. \end{aligned} \tag{2.54}$$

In the second line we've made the substitution $u = t + s\cos 2\theta$ and $v = s\sin 2\theta$, and $A \subset \mathbb{R}^2$ is the region $v < u\tan\theta$. This integral can be performed directly, giving a contribution

$$k_V(z) = \frac{c^{ab} t_{a,V} \tau_V t_{b,V}}{4z}\,, \tag{2.55}$$

to the classical k-matrix. This is independent of the angle of incidence θ, as expected from diffeomorphism invariance in the Σ direction.[4] Note that the individual contribution from each of the three diagrams in figure 2.6 is θ dependent.

To order $\hbar$, we obtain a quasi-classical K-matrix

$$K(z) = \tau_V + \frac{\hbar}{4z}\,(t_a \tau t^a)_V + \mathcal{O}(\hbar^2)\,. \tag{2.56}$$

We now consider including a boundary line operator, which classically lives in a representation W of $\mathfrak{h}$. In addition to the Feynman diagrams of the previous section, there are two new tree level contributions to the k-matrix arising from interactions between the bulk and boundary Wilson lines:

Fig. 2.8 Coupling between reflecting and boundary Wilson lines

Rewriting this configuration on the orbifold we can see that both of these contributions come from a single diagram.

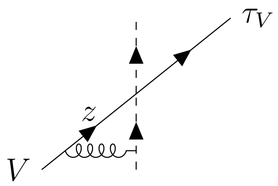


Fig. 2.9 Coupling between a bulk and boundary Wilson line on the orbifold

Since the gauge field along L is restricted to lie in $\mathfrak{h}$, the boundary line couples as

$$b_{\alpha,W} \int_L A_\Sigma^\alpha\,. \tag{2.57}$$

To determine the contribution of the diagram 2.9 we first note that one end of the propagator is attached to the boundary Wilson line. In this circumstance, it can

[4]It is not actually the case that the VEV of a straight reflecting Wilson line is independent of the angle of incidence, θ. It is, however, true at order $\hbar$. We observed a similar behaviour for crossing Wilson lines in the bulk in subsection 1.3.3, and will consider this effect on the orbifold in subsection 2.3.3.

be simplified considerably. Assuming that $v_2 \in L$, which by definition means that $\mathcal{P}v_2 = v_2$, we find

$$\widetilde{P}_{iy}(v_1, v_2) = c\Delta_{iy}(v_1 - v_2) + c^\sigma \mathbb{P}_y{}^y \Delta_{iy}(v_1 - v_2) = (c + c^\sigma)\Delta_{iy}(v_1 - v_2)\,. \tag{2.58}$$

Writing $c = t_\alpha \otimes t^\alpha + t_\mu \otimes t^\mu$, we have $c^\sigma = t_\alpha \otimes t^\alpha - t_\mu \otimes t^\mu$. Therefore $c + c^\sigma = 2c^{\mathfrak{h}}$ for $c^{\mathfrak{h}} = t_\alpha \otimes t^\alpha$ the projection of the split Casimir onto $\mathfrak{h} \otimes \mathfrak{h}$. This allows us to replace the orbifold propagator with twice the bulk propagator, albeit with a slightly modified colour factor. From the bulk result for the classical r-matrix in section 1.3.3 we deduce that diagram 2.9 contributes

$$\delta k_{V\otimes W}(z) = \frac{2}{z} c^{\alpha\beta} t_{\alpha,V}\tau_V \otimes t_{\beta,W}\,. \tag{2.59}$$

Combining this with equation (2.55) we obtain the more general quasi-classical K-matrix

$$K(z) = \tau_V \otimes \mathbb{1}_W + \frac{\hbar}{4z}(t_a\tau t^a)_V \otimes \mathbb{1}_W + \frac{2\hbar}{z}(t^\alpha\tau)_V \otimes b_{\alpha,W} + \mathcal{O}(\hbar^2)\,, \tag{2.60}$$

which has associated abstract classical ℓ-matrix

$$\ell(z) = k(z)(\tau^{-1} \otimes 1) = \frac{1}{4z} t_a \sigma(t^a) \otimes 1 + \frac{2}{z} t^\alpha \otimes b_\alpha\,. \tag{2.61}$$

This formula gives the asymptotic behaviour of a rational K-matrix at first order in $\hbar$.

We verify in the next section that this is indeed a solution to the CBYBE with boundary degrees of freedom. The fact that perturbation theory allows one systematically to construct such explicit expressions, even for arbitrary $\dim V$ and $\dim W$, is a key virtue of the gauge theory approach. We do not believe this expression for the quasi-classical K-matrix has appeared in the literature before, though it can be derived from the intertwiner relation of Delius, MacKay and Short [61, 113].

2.2.5 Proof that the rational ℓ-matrix obeys the CBYBE

In this section we prove that the rational ℓ-matrix obtained from gauge theory does indeed obey the CBYBE given in equation (2.10). This is equivalent to showing that the corresponding quasi-classical K-matrix obeys the BYBE up to order $\hbar^2$.

Proof. We begin by writing

$$\ell(z) = \frac{1}{4z} t_a \sigma(t^a) \otimes 1 + \frac{2}{z} t^\alpha \otimes b_\alpha = \ell_0(z) \otimes 1 + \delta\ell(z) \,, \tag{2.62}$$

where $\ell_0(z)$ is generated by the self-interaction of a bulk Wilson line, and $\delta\ell(z)$ by the coupling between bulk and boundary Wilson lines. This allows us to split the proof into two parts: we begin by showing that $\ell_0(z)$ obeys the CBYBE without boundary degrees of freedom, and then show that adding $\delta\ell(z)$ leads to a solution with boundary degrees of freedom.

Recall equation (2.6), the CBYBE without boundary degrees of freedom:

$$\begin{aligned} 0 = {} & [r(z_1 - z_2), \sigma_2 r_{12}(z_1 + z_2)] \\ & + [r_{12}(z_1 - z_2) - \sigma_2 r_{12}(z_1 + z_2), \ell_1(z_1)] \\ & \quad + [r_{12}(z_1 - z_2) + \sigma_2 r_{12}(z_1 + z_2), \ell_2(z_2)] \,. \end{aligned} \tag{2.63}$$

We take the classical r-matrix to be

$$r(z) = \frac{1}{z}\, t_a \otimes t^a \,. \tag{2.64}$$

and for now take $\ell(z) = \ell_0(z)$. Note that

$$[r_{12}(z_1 - z_2), \ell_1(z_1)] = \frac{1}{4z_1(z_1 - z_2)} [t^a, t^b \sigma(t_b)] \otimes t^a \,. \tag{2.65}$$

The commutator on the right hand side can be simplified using

$$\begin{aligned} [t^a, t^b\sigma(t_b)] &= [t^a, t^b]\sigma(t_b) + t_b[t^a, \sigma(t^b)] = [t^a, t^b]\sigma(t_b) + \sigma(t_b)[t^a, t^b] \\ &= f^{abc}\{\sigma(t_b), t_c\} = -f^{abc}\{t_b, \sigma(t_c)\} \,. \end{aligned} \tag{2.66}$$

Here the curly brackets $\{\cdot, \cdot\}$ refer to the anticommutator. In the second equality we've made use of the σ-invariance of the split Casimir, i.e., $c = \sigma_1\sigma_2 c$. Therefore

$$[r_{12}(z_1 - z_2), \ell_1(z_1)] = -\frac{f^{abc}}{4z_1(z_1 - z_2)} \{t_b, \sigma(t_c)\} \otimes t_a \,. \tag{2.67}$$

An almost identical manipulation applied to $[\sigma_2 r_{12}(z_1+z_2), \ell_1(z)]$ shows that

$$\begin{aligned}[r_{12}(z_1-z_2)-\sigma_2 r_{12}(z_1+z_2), \ell_1(z_1)] &= -\frac{f^{abc}}{4z_1}\left(\frac{1}{z_1-z_2}+\frac{1}{z_1+z_2}\right)\{t_b,\sigma(t_c)\}\otimes t_a \\ &= -\frac{f^{abc}}{2(z_1-z_2)(z_1+z_2)}\{t_b,\sigma(t_c)\}\otimes t_a\,.\end{aligned} \tag{2.68}$$

Similarly

$$[r_{12}(z_1-z_2)+\sigma_2 r_{12}(z_1+z_2), \ell_2(z_2)] = -\frac{f^{abc}}{2(z_1-z_2)(z_1+z_2)}t_a\otimes\{t_b,\sigma(t_c)\}\,. \tag{2.69}$$

Finally the term involving only the classical r-matrix is

$$[r_{12}(z_1-z_2), \sigma_2 r_{12}(z_1+z_2)] = \frac{1}{(z_1-z_2)(z_1+z_2)}(t_b t_c\otimes t^b\sigma(t^c) - t_c t_b\otimes\sigma(t^c)t^b)\,. \tag{2.70}$$

Substituting (2.68), (2.69) & (2.70) into the right hand side of (2.63) gives

$$\frac{2t_b t_c\otimes t^b\sigma(t^c) - 2t_c t_b\otimes\sigma(t^c)t^b - f^{abc}\{t_b,\sigma(t_c)\}\otimes t_a - f^{abc}t_a\otimes\{t_b,\sigma(t_c)\}}{2(z_1-z_2)(z_1+z_2)}\,. \tag{2.71}$$

We need to verify that the numerator vanishes. To that end, we write

$$\begin{aligned}t_b t_c\otimes t^b\sigma(t^c) - t_c t_b\otimes\sigma(t^c)t^b &= t_b t_c\otimes[t^b,\sigma(t^c)] + [t^b,t^c]\otimes\sigma(t_c)t_b \\ &= f^{abc}(t_b\sigma(t_c)\otimes t_a + t_a\otimes\sigma(t_c)t_b)\,,\end{aligned} \tag{2.72}$$

and substituting this into the numerator of (2.71) gives

$$\begin{aligned}0 &= f^{abc}\Big((2t_b\sigma(t_c)-\{t_b,\sigma(t_c)\})\otimes t_a + t_a\otimes(2\sigma(t_c)t_b-\{t_b,\sigma(t_c)\})\Big) \\ &= f^{abc}([t_b,\sigma(t_c)]\otimes t_a + t_a\otimes[\sigma(t_c),t_b]) = [t_b,\sigma(t_c)]\otimes[t^b,t^c] + [t^b,t^c]\otimes[\sigma(t_c),t_b]\,.\end{aligned} \tag{2.73}$$

The final two terms on the right hand side cancel one another, and so $\ell_0(z)$ obeys the CBYBE without boundary degrees of freedom.

Now we need to show that replacing $\ell_0(z)$ by $\ell_0(z)\otimes 1 + \delta\ell(z)$ gives a solution to the CBYBE with boundary degrees of freedom.

Recall equation (2.10), the CBYBE with boundary degrees of freedom:

$$\begin{aligned}0 &= [r(z_1 - z_2), \sigma_2 r_{12}(z_1 + z_2)] + [\ell_1(z_1), \ell_2(z_2)] \\ &\quad + [r_{12}(z_1 - z_2) - \sigma_2 r_{12}(z_1 + z_2), \ell_1(z_1)] \\ &\quad\quad + [r_{12}(z_1 - z_2) + \sigma_2 r_{12}(z_1 + z_2), \ell_2(z_2)]\,.\end{aligned} \tag{2.74}$$

Substituting the classical r-matrix and $\ell(z) = \ell_0(z) \otimes 1 + \delta\ell(z)$ into the above we find that there are no terms involving both $\ell_0(z)$ and $\delta\ell(z)$. (This is simply an artefact of working at first order $\hbar$. At higher order it is not even possible to split the K-matrix in this way.) Assuming the CBYBE without boundary degrees of freedom holds for $\ell_0(z)$ the terms that remain are

$$\begin{aligned}0 &= [\delta\ell_1(z_1), \delta\ell_2(z_2)] \\ &\quad + [r_{12}(z_1 - z_2), \delta\ell_1(z_1) + \delta\ell_2(z_2)] \\ &\quad\quad - [\sigma_2 r_{12}(z_1 + z_2), \delta\ell_1(z_1) - \delta\ell_2(z_2)]\,.\end{aligned} \tag{2.75}$$

Note that

$$\begin{aligned}[r_{12}(z_1 - z_2), \delta\ell_1(z_1)] &= \frac{2}{z_1(z_1 - z_2)}[t^b, t^\gamma] \otimes t_b \otimes b_\gamma \\ &= 2\frac{f^{ab\gamma}}{z_1(z_1 - z_2)} t_a \otimes t_b \otimes b_\gamma\,.\end{aligned} \tag{2.76}$$

An identical manipulation applied to $[r_{12}(z_1 - z_2), \delta\ell_2(z_2)]$ shows that

$$\begin{aligned}[r_{12}(z_1 - z_2), \delta\ell_1(z_1) + \delta\ell_2(z_2)] &= \frac{2f^{ab\gamma}}{z_1 - z_2}\left(\frac{1}{z_1} - \frac{1}{z_2}\right) t_a \otimes t_b \otimes b_\gamma \\ &= -\frac{2f^{ab\gamma}}{z_1 z_2} t_a \otimes t_b \otimes b_\gamma\,, \\ &= -\frac{2f^{\alpha\beta\gamma}}{z_1 z_2} t_\alpha \otimes t_\beta \otimes b_\gamma - \frac{2f^{\mu\nu\gamma}}{z_1 z_2} t_\mu \otimes t_\nu \otimes b_\gamma\,.\end{aligned} \tag{2.77}$$

Similarly

$$\begin{aligned}[\sigma_2 r_{12}(z_1 + z_2), \delta\ell_1(z_1) - \delta\ell_2(z_2)] &= \frac{2f^{ab\gamma}}{z_1 z_2} t_a \otimes \sigma(t_b) \otimes b_\gamma\,, \\ &= -\frac{2f^{\alpha\beta\gamma}}{z_1 z_2} t_\alpha \otimes t_\beta \otimes b_\gamma + \frac{2f^{\mu\nu\gamma}}{z_1 z_2} t_\mu \otimes t_\nu \otimes b_\gamma\,.\end{aligned} \tag{2.78}$$

Finally we have

$$\begin{aligned}[\ell_1(z_1), \ell_2(z_2)] &= \frac{4}{z_1 z_2} t_\alpha \otimes t_\beta \otimes [b^\alpha, b^\beta] \\ &= \frac{4 f^{\alpha\beta\gamma}}{z_1 z_2} t_\alpha \otimes t_\beta \otimes b_\gamma\end{aligned} \tag{2.79}$$

From (2.77), (2.78) & (2.79) it is clear that (2.75) holds, and so $\ell(z)$ obeys the CBYBE with boundary degrees of freedom. □

2.2.6 Examples of rational K-matrices with $W \cong \mathbb{C}$

K-matrices are most often studied in the case where there are no boundary degrees of freedom. It is tempting to imagine that these should be obtained as the VEV of a reflecting Wilson line without the insertion of a boundary Wilson line. This is not quite correct, since even for $\mathfrak{g}$ simple the subalgebra $\mathfrak{h}$ may have a non-trivial centre. By introducing boundary Wilson lines in 1 dimensional representations of this centre we still obtain K-matrices with apparently no boundary degrees of freedom. This observation is essential to obtain the full parameter space of solutions to the BYBE.

Suppose $\mathfrak{h}$ does have a non-trivial centre, and let Q be its generator. (For all symmetric spaces the centre of $\mathfrak{h}$ is at most 1 dimensional.) Then the 1 dimensional representations of $\mathfrak{h}$ are labelled by $Q \mapsto q \in \mathbb{C}$ with all other generators vanishing. Our K-matrices are then parametrized by q, which can be viewed physically as the charge of the boundary Wilson line. In this case, equation (2.60) becomes

$$K(z) = \tau_V + \frac{\hbar}{z}\left(\frac{1}{4}(t_a \tau t^a)_V + \frac{2q}{\operatorname{tr} Q^2}(Q\tau)_V\right) + \mathcal{O}(\hbar^2)\,, \tag{2.80}$$

where we have identified $V \otimes \mathbb{C} \cong V$. If $\mathfrak{h}$ has no centre then we set $q = 0$.

We now explicitly evaluate the above formula in a few simple cases, and match the results to known examples of K-matrices. Since the notion of a quasi-classical K-matrix was first introduced by the author in [29], all K-matrices appearing in the literature are not formal power series in $\hbar$ but instead depend on finite parameters. Hereafter we shall refer to these as finite K-matrices. Fortunately, in the rational case there is canonical way to introduce $\hbar$ dependence so that the result is invariant under simultaneous rescalings of $\hbar$ and z. We simply expand a finite K-matrix in a Laurent series around $z = 0$, perhaps after rescaling so that it has a sensible $z \to \infty$ limit, and then replace $1/z \mapsto \hbar/z$.

Although we have only determined the first order contribution to the quasi-classical K-matrix generated by CS_4, exploiting a proof of uniqueness in subsection 2.3.5 will allow us to extend our results to all orders in $\hbar$.

The examples we consider have been chosen to capture the features of a generic solution. In each we take $\mathfrak{g}$ to be a classical Lie algebra and $V = U$, the defining vector representation.[5]

Example 2.2.1. Let $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_{n-k}(\mathbb{C}) \oplus \mathfrak{sl}_k(\mathbb{C}) \oplus \mathbb{C})$ and $\tau_U = e^{i\pi k/n}\mathrm{diag}(\mathbb{1}_{n-k}, -\mathbb{1}_k)$, for $k = 1, \dots, \lfloor n/2 \rfloor$. Taking the representation W to be $Q = \mathrm{diag}(k\mathbb{1}_{n-k}, -(n-k)\mathbb{1}_k)/n \mapsto q$, our solution gives

$$K(z) = \tau_U + \frac{\hbar}{z}\left(\left(\frac{nq}{k(n-k)} + \frac{n-2k}{4}\right)e^{i\pi k/n}\mathbb{1}_U - \left(\frac{(n-2k)q}{k(n-k)} + \frac{1}{4n}\right)\tau_U\right) + \mathcal{O}(\hbar^2)\,. \tag{2.81}$$

Introducing

$$\lambda = \left(\frac{nq}{k(n-k)} + \frac{n-2k}{4}\right)e^{i\pi k/n}\,, \tag{2.82}$$

this is consistent with

$$K(z) \sim \tau_U + \frac{\hbar\lambda}{z}\mathbb{1}_U \tag{2.83}$$

to first order in $\hbar$. (Recall that $K_1(z) \sim K_2(z)$ if $K_1(z) = G(z,\hbar)K_2(z)$ for $G = 1 + \mathcal{O}(\hbar)$ a formal power series in $\hbar$ with values in the ring of meromorphic functions on $\mathbb{C}$.) Note that $\lambda \in \mathbb{C}$ is a free parameter. The K-matrix appearing on the right hand side is the motivating example from section 2.1 and originally introduced in [39]. It is the most general solution to the BYBE for Yang's rational R-matrix [125, 7].

In subsection 2.3.5 we prove that, up to scale, there is a unique quasi-classical K-matrix with classical ℓ-matrix consistent with equation (2.81). This proves that equation (2.83) holds to all orders in $\hbar$ with λ related to q by (2.82).[6]

Curiously, the right hand side of (2.83) is in a sense independent of $\hbar$. In particular, rescaling $\hbar \mapsto t\hbar$ can be compensated in the K-matrix by rescaling $\lambda \mapsto t^{-1}\lambda$. The next example demonstrates that this is not a generic feature of quasi-classical K-matrices, but in chapter 3 we will observe similar behaviour for trigonometric and elliptic K-

[5] Recall that the defining vector representation of a classical simple Lie algebra always lifts to the associated Yangian. For further details see [35]. Also, the 1 dimensional representations of $\mathfrak{h}$ lift to the twisted Yangian [7, 24, 85, 86].

[6] λ cannot receive quantum corrections due to the symmetry of the theory under simultaneously rescaling z and $\hbar$.

matrices associated to the fundamental representation of $\mathfrak{sl}_n(\mathbb{C})$. We have been unable to find an explanation for this property in the gauge theory.

Example 2.2.2. Let $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{so}_n(\mathbb{C}), \mathfrak{so}_{n-k}(\mathbb{C})\oplus\mathfrak{so}_k(\mathbb{C}))$ and $\tau_U = (-)^k\mathrm{diag}(\mathbb{1}_{2n+1-k}, -\mathbb{1}_k)$, for $n > 2$, $n \neq 4$ and $k = 1, \ldots, \lfloor n/2 \rfloor$. For $k > 2$, $\mathfrak{h}$ has trivial centre and so our solution gives

$$K(z) = \tau_U + \frac{\hbar}{4z}\Big((n-2k)(-)^k\mathbb{1}_U - \tau_U\Big) + \mathcal{O}(\hbar^2)\,. \tag{2.84}$$

which is consistent with

$$K(z) \sim \tau_U + \frac{\hbar(n-2k)}{4z}(-)^k\mathbb{1}_U \tag{2.85}$$

to first order in $\hbar$. The K-matrix on the right hand side is the one appearing in [6, 112, 113]. Again, our uniqueness result in section 2.3.5 shows that equation (2.85) holds to all orders in $\hbar$. It has no continuous parameters independent of those in the R-matrix, and so is not 'independent' of $\hbar$ in the sense introduced above.

For n even and k odd, $\mathfrak{h}$ is associated to an outer automorphism of $\mathfrak{so}_n(\mathbb{C})$ realised as conjugation by an element of $\mathrm{O}_n(\mathbb{C})$ with determinant -1. (Note that $\det\tau_U = -1$.) Since this automorphism preserves the defining vector representation of $\mathfrak{so}_n(\mathbb{C})$, equation (2.60) still applies.

When $k = 2$, $\mathfrak{h}$ contains a copy of $\mathfrak{so}_2(\mathbb{C}) \cong \mathbb{C}$. Our solution is therefore enhanced to

$$K(z) = \tau_U + \frac{\hbar}{4z}\Big((n-4)(-)^k\mathbb{1}_U - \tau_U + 8q(-)^kQ_U\Big) + \mathcal{O}(\hbar^2)\,, \tag{2.86}$$

where

$$Q_U = \begin{pmatrix} 0_{n-2} & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}. \tag{2.87}$$

Viewing Q as an element of $\mathfrak{so}_n(\mathbb{C})$, the representation W is $Q \mapsto q$. This is consistent with

$$K(z) \sim \tau_U + \frac{\hbar(n-4)}{4z}(-)^{km}\mathbb{1}_U + \frac{2\hbar q}{z}(-)^{km}Q_U \tag{2.88}$$

to first order in $\hbar$. The K-matrix on the right hand side appears in [6, 112, 113]. Uniqueness guarantees that this holds to all orders in $\hbar$.

This example illustrates that even when $\mathfrak{h}$ has an abelian summand the corresponding family of K-matrices is not necessarily 'independent' of $\hbar$.

Example 2.2.3. For completeness we include the remaining conjugacy class of involutive inner automorphisms of $\mathfrak{so}_{2n}(\mathbb{C})$, given by $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}_{2n}(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C})$ and

$$\tau_U = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{2.89}$$

for $n \neq 1, 2$. Here $\mathfrak{h}$ has an abelian summand, and so

$$K(z) = \tau_V + \frac{\hbar}{4z}\Big(\tau_U + 4q\mathbb{1}_U\Big) + \mathcal{O}(\hbar^2) \tag{2.90}$$

where the representation W is

$$Q = \frac{1}{n}\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \mapsto q\,. \tag{2.91}$$

This is consistent with

$$K(z) \sim \tau_U + \frac{\hbar q}{z}\mathbb{1}_U \tag{2.92}$$

to first order in $\hbar$. The K-matrix on the right hand side appears in [6, 112, 113]. By uniqueness this holds to all orders in $\hbar$.

This has may of the same features as example 2.2.1. In particular, it is ‘independent’ of $\hbar$.

Between the K-matrices generated in examples 2.2.2 & 2.2.3, we have recovered quasi-classical counterparts of all solutions to the BYBE for the standard rational R-matrix associated to the fundamental of $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$.

We do not include examples for $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, since these are qualitatively the same as for $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$. Needless to say, our formula (2.80) together with the uniqueness result in subsection 2.3.5 reproduces the K-matrices of [6, 112, 113]. For further details see [29].

We have not considered the trivial involutive automorphism $\sigma = \mathrm{id.}$ corresponding to $\mathfrak{g} = \mathfrak{h}$. This is because the theory has a global G symmetry, and so by Schur’s lemma the K-matrix associated to any irreducible representation of $\mathfrak{g}$ must be proportional to the identity. $\sigma = \mathrm{id.}$ should nonetheless lead to non-trivial K-matrices with boundary degrees of freedom, and there is an associated twisted Yangian [24].

2.3 Twisted Yangians from CS_4

In this section we discuss the emergence of the twisted Yangian in CS_4 on $\widetilde{\mathcal{V}}$. We assume familiarity with the content of subsection 1.3.4.

2.3.1 Bulk-boundary OPE

We have reviewed in subsection 1.3.4 how the Yangian emerges from CS_4 theory on $\mathcal{V}$. In particular, there is a direct relation between the OPE of line operators in the gauge theory and the Yangian coproduct. Here we perform an analogous computation for line operators on $\widetilde{\mathcal{V}}$, showing how the coideal structure of the twisted Yangian arises from quantum contributions to the OPE between a bulk and boundary line operator.

Consider a bulk and boundary Wilson line in representations V and W of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. We take the bulk Wilson line to be supported at $z = 0$, the same point in $C = \mathbb{C}$ at which the boundary Wilson line is supported. We then bring the bulk Wilson line to the boundary in parallel. Classically, as the bulk line operator approaches L we generate a new boundary Wilson line in the tensor product $V \otimes W$, where the representation V is restricted to $\mathfrak{h} \subset \mathfrak{g}$ since $\iota_L^* A^{\mathfrak{m}} = 0$. As in the bulk, this OPE receives a quantum correction. At $\mathcal{O}(\hbar)$ the correction comes from the two Feynman diagrams below.

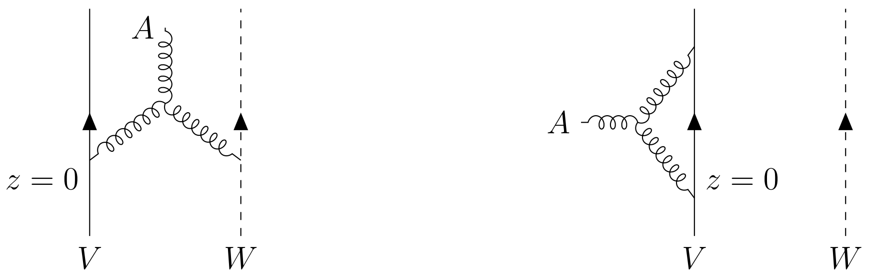


Fig. 2.10 Quantum corrections to the bulk-boundary OPE

The diagram on the left is completely analogous to that of the bulk OPE, except that we must compute it using the orbifold propagator $\widetilde{P}$. The second diagram might appear to simply renormalize the coupling of the bulk Wilson line, an effect already present and accounted for. However, as this line approaches L, the diagram receives further contributions from the 'image' part of the propagator, an effect not present when the bulk Wilson line is well separated from the boundary.

Let's start by calculating the contribution from the first diagram, which is

$$\frac{i\hbar}{4\pi} f_{abc} t_{d,V} \otimes b_{\varepsilon,W} \int_{-\infty}^{\infty} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}t \int_{\mathcal{V}} \mathrm{d}^4 v\, A_y^a(v) \Big(\widetilde{P}_{\bar{z}y}^{bd}(v,\gamma(s)) \widetilde{P}_{xy}^{c\varepsilon}(v,\lambda(t)) - \widetilde{P}_{xy}^{bd}\big(v,\gamma(s)\big) \widetilde{P}_{\bar{z}y}^{c\varepsilon}(v,\lambda(t)) \Big) . \tag{2.93}$$

Here $\gamma(s)$ and $\lambda(t)$ are the paths of the bulk and boundary Wilson lines respectively, given explicitly by

$$\gamma(s) = (-\epsilon, s, 0, 0)\,, \qquad \lambda(t) = (0, t, 0, 0) \tag{2.94}$$

where ϵ is the separation between the two lines. Eventually we'll take the $\epsilon \to 0$ limit. From (2.58), we know that a propagator connected at one end to a point $v_2 \in L$ simplifies to

$$\widetilde{P}_{iy}^{c\varepsilon}(v_1, v_2) = 2c^{\mathfrak{h}} \Delta_{iy}(v_1, v_2) \tag{2.95}$$

for $c^{\mathfrak{h}}$ the projection of the split Casimir onto $\mathfrak{h} \otimes \mathfrak{h}$. With this understanding, the index c can be replaced by γ, whereupon the bracketed term in the integrand becomes

$$\frac{c^{\gamma\epsilon}}{\pi(x^2 + (y-t)^2 + |z|^2)^2} \Big(2\bar{z} \widetilde{P}_{\bar{z}y}^{bd}(v,\gamma(s)) + x \widetilde{P}_{xy}^{bd}(v,\gamma(s)) \Big) . \tag{2.96}$$

Substituting the explicit form of the propagators (2.49) into the above and performing the s and t integrals gives

$$\frac{\epsilon \bar{z} c^{\gamma\epsilon}}{4(x^2 + |z|^2)^{3/2}} \left[\frac{c^{bd}}{((x-\epsilon)^2 + |z|^2)^{3/2}} - \frac{(c^{\sigma})^{bd}}{((x+\epsilon)^2 + |z|^2)^{3/2}} \right] . \tag{2.97}$$

Now, noting that $c + c^{\sigma} \in \mathfrak{h}^{\otimes 2}$, and $c - c^{\sigma} \in \mathfrak{m}^{\otimes 2}$, we can write the contribution of the first diagram as

$$\frac{i\hbar}{16\pi} \int_{\mathcal{V}} \mathrm{d}^4 v\, A_y^a(v) \Big(f_a{}^{\beta\gamma} t_{\beta,V} \otimes b_{\gamma,W} I_-(x,z,\bar{z},\epsilon) + f_a{}^{\nu\gamma} t_{\nu,V} \otimes b_{\gamma,W} I_+(x,z,\bar{z},\epsilon) \Big) , \tag{2.98}$$

where we've introduced the functions

$$I_{\pm}(x,z,\bar{z},\epsilon) = \frac{\epsilon \bar{z}}{(x^2 + |z|^2)^{3/2}} \left(\frac{1}{((x-\epsilon)^2 + |z|^2)^{3/2}} \pm \frac{1}{((x+\epsilon)^2 + |z|^2)^{3/2}} \right) . \tag{2.99}$$

We now wish to take the limit $\epsilon \to 0$. Certainly $I_{\pm}$ converge uniformly to 0 on the complement of any neighbourhood of L, but are unbounded for sufficiently small ϵ inside any such neighbourhood. This suggests that in this limit these functions tend to distributions supported on L, and constant along L since they are independent of y. By dimensional analysis and rotation symmetry in the z direction, this distribution must be proportional to $\partial_z \delta^3(x,z,\bar{z})$ where $\delta^3(x,z,\bar{z})$ is a δ-function normalised so that $\int_{\mathbb{R}\times\mathbb{C}} \mathrm{d}x\, \mathrm{d}^2z\, \delta^3(x,z,\bar{z}) = 1$. This allows us to write $\lim_{\epsilon\to 0} I_\pm(x,z,\bar{z},\epsilon) = -\lambda_\pm \partial_z \delta^3(x,z,\bar{z})$, where

$$\lambda_\pm = \int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbb{C}} \mathrm{d}^2z\, z I_\pm(x,z,\bar{z},1)\,. \tag{2.100}$$

Since $I_-(x,z,\bar{z},\epsilon)$ is antisymmetric under $x \mapsto -x$, by symmetry $\lambda_- = 0$. We determine the constant λ_+ in appendix A.1 to be $-16\pi i$. Substituting this value in (2.98) the contribution (2.100) simplifies to

$$-\hbar f_\mu{}^{\nu\gamma} t_{\nu,V} \otimes b_{\gamma,W} \int_{-\infty}^{\infty} \mathrm{d}y\, \partial_z A_y^\mu(\lambda(y))\,, \tag{2.101}$$

where we've used the property $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$ of a symmetric pair to simplify the structure constants.

Let's now consider the second Feynman diagram, which contributes

$$\frac{i\hbar}{4\pi} f_{abc} t_{d,V} t_{e,V} \otimes \mathbb{1}_W \int_{-\infty<s<t<\infty} \mathrm{d}s\, \mathrm{d}t \int_{\mathcal{V}} \mathrm{d}^4v\, A_y^a(v) \Big(\widetilde{P}^{bd}_{\bar{z}y}(v,\gamma(s)) \widetilde{P}^{ce}_{xy}(v,\gamma(t)) - \widetilde{P}^{bd}_{xy}(v,\gamma(s)) \widetilde{P}^{ce}_{\bar{z}y}(v,\gamma(t)) \Big) \tag{2.102}$$

with $\gamma(s) = (-\epsilon, s, 0, 0)$ as before. The combination of propagators appearing in the integrand is

$$\begin{aligned} &\widetilde{P}^{bd}_{\bar{z}y}(v,\gamma(s)) \widetilde{P}^{ce}_{xy}(v,\gamma(t)) - \widetilde{P}^{bd}_{xy}(v,\gamma(s)) \widetilde{P}^{ce}_{\bar{z}y}(v,\gamma(t)) \\ &\quad = \frac{\epsilon \bar{z}}{\pi^2} \Bigg(\frac{(c^\sigma)^{bd}\, c^{ce}}{((x-\epsilon)^2+(y-s)^2+|z|^2)^2((x+\epsilon)^2+(y-t)^2+|z|^2)^2} \\ &\qquad - \frac{c^{bd}\,(c^\sigma)^{ce}}{((x+\epsilon)^2+(y-s)^2+|z|^2)^2((x-\epsilon)^2+(y-t)^2+|z|^2)^2} \Bigg)\,. \end{aligned} \tag{2.103}$$

As before, we decompose this according to the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with $b \to (\beta,\mu)$, $c \to (\gamma,\nu)$, $d \to (\delta,\rho)$ and $e \to (\varepsilon,\sigma)$. This allows us to write equation (2.103) as

$$\Big(c^{\beta\delta}c^{\gamma\varepsilon} - c^{\mu\rho}c^{\nu\sigma}\Big) J_-(x,z,\bar{z},s-y,t-y,\epsilon) + \Big(c^{\beta\delta}c^{\nu\sigma} - c^{\mu\rho}c^{\gamma\varepsilon}\Big) J_+(x,z,\bar{z},s-y,t-y,\epsilon)\,, \tag{2.104}$$

where we have defined

$$J_{\pm}(x,z,\bar{z},s,t,\epsilon)=\frac{\epsilon\bar{z}}{\pi^2}\left(\frac{1}{((x-\epsilon)^2+s^2+|z|^2)^2((x+\epsilon)^2+t^2+|z|^2)^2}\pm(s\leftrightarrow t)\right). \tag{2.105}$$

Since the gauge field is independent of s, t, we can integrate over these variables directly. One finds

$$\int_{-\infty<s<t<\infty}\mathrm{d}s\,\mathrm{d}t\,J_{-}(x,z,\bar{z},s,t,\epsilon)=0 \tag{2.106}$$

using the fact that J_- is antisymmetric under exchange of s and t, and invariant under $s\mapsto -s$, $t\mapsto -t$. On the other hand, the integral of J_+ is non-vanishing, and we find

$$\begin{aligned}K_{+}(x,z,\bar{z},\epsilon)&=\int_{-\infty<s<t<\infty}\mathrm{d}s\,\mathrm{d}t\,J_{+}(x,z,\bar{z},s,t,\epsilon)\\&=\frac{\epsilon\bar{z}}{4((x-\epsilon)^2+|z|^2)^{3/2}((x+\epsilon)^2+|z|^2)^{3/2}}\,.\end{aligned} \tag{2.107}$$

We now take the $\epsilon\to 0$ limit. By an identical argument to that used for the first diagram, we deduce that

$$\lim_{\epsilon\to 0}K_{+}(x,z,\bar{z},\epsilon)=-\mu_{+}\partial_z\delta^3(x,z,\bar{z})\,, \tag{2.108}$$

where, as shown in appendix A.1, the constant

$$\mu_{+}=\int_{-\infty}^{\infty}\mathrm{d}x\int_{\mathbb{C}}\mathrm{d}^2z\,zK_{+}(x,z,\bar{z},1)=-\pi i\,. \tag{2.109}$$

Therefore the contribution of the second diagram to the OPE is

$$\frac{\hbar}{4}\Big(f_{a\beta\nu}c^{\beta\delta}c^{\nu\sigma}t_{\delta,V}t_{\sigma,V}-f_{a\mu\gamma}c^{\mu\rho}c^{\gamma\varepsilon}t_{\rho,V}t_{\varepsilon,V}\Big)\otimes\mathbb{1}_W\int_{-\infty}^{\infty}\mathrm{d}y\,\partial_zA^a_y(\ell(y))\,. \tag{2.110}$$

Since $[\mathfrak{h},\mathfrak{m}]\subseteq\mathfrak{m}$, the structure constants appearing here are non-zero only when the index a takes values in $\mathfrak{m}$. The contribution of the second Feynman diagram is then

$$\frac{\hbar}{4}f_{\mu}{}^{\beta\nu}\{t_{\beta,V},t_{\nu,V}\}\otimes\mathbb{1}_W\int_{-\infty}^{\infty}\mathrm{d}y\,\partial_zA^{\mu}_y(\ell(y))\,. \tag{2.111}$$

Recall that the curly brackets $\{\cdot,\cdot\}$ refer to the anticommutator. Combining the two diagrams, the total contribution to the OPE between a bulk and boundary Wilson line at first order in $\hbar$ is

$$\left(-\hbar f_{\mu}{}^{\nu\alpha}t_{\nu,V}\otimes b_{\alpha,W}+\frac{\hbar}{4}f_{\mu}{}^{\beta\nu}\{t_{\beta},t_{\nu}\}_V\otimes\mathbb{1}_W\right)\int_L\partial_zA^{\mu}_{\Sigma}\,. \tag{2.112}$$

As in section 1.3.4, the presence of z derivatives of A in (2.112) means that we've obtained an operator which is not an ordinary boundary Wilson line. Instead, at least classically, it is one of the generalised boundary line operators introduced in section 2.2.2. To obtain an operation which is closed we should instead compute the OPE of a generalised bulk and boundary line operator in representations V and W. At first order in $\hbar$ the only novelty is that the external gauge field can couple directly to $t^{(1)}_{a,V}$ or $b^{(1)}_{\mu,W}$ on the bulk and boundary operator respectively. The OPE is therefore a generalised boundary line operator with

$$\begin{aligned} b^{(0)}_{\alpha,V\otimes W} &= t^{(0)}_{\alpha,V} \otimes \mathbb{1}_W + \mathbb{1}_V \otimes b^{(0)}_{\alpha,W} \,, \\ b^{(1)}_{\mu,V\otimes W} &= \left(t^{(1)}_{\mu,V} + \frac{\hbar}{4} f_\mu{}^{\beta\nu} \{ t^{(0)}_{\beta,V}, t^{(0)}_{\nu,V} \} \right) \otimes \mathbb{1}_W + \mathbb{1}_V \otimes b^{(1)}_{\mu,W} - \hbar f_\mu{}^{\nu\alpha} t^{(0)}_{\nu,V} \otimes b^{(0)}_{\alpha,W} \,. \end{aligned} \tag{2.113}$$

(We continue to write $V \otimes W$ for the representation of the resulting boundary line operator.) In the next section we discuss how this quantum correction necessitates a deformation of the algebra obeyed by the $\{b^{(m)}_\bullet\}_{m\in\mathbb{Z}^+_0}$.

2.3.2 Twisted Yangian

In the previous section we have seen how the OPE of bulk and boundary line operators receives a quantum correction.

To interpret this abstractly we first need to understand the classical OPE. It takes a bulk and boundary line operator in representations V and W of $\mathfrak{g}[[u]]$ and $\mathfrak{g}^\sigma[[u]]$ respectively, and returns a boundary line operator in the tensor product representation $V \otimes W$ of $\mathfrak{g}^\sigma[[u]]$ defined by (2.113) with $\hbar = 0$. Abstractly, the tensor product representation is induced by a coideal structure on $U(\mathfrak{g}^\sigma[[u]])$ as a subalgebra of $U(\mathfrak{g}[[u]])$, i.e., a morphism of associative algebras

$$\begin{aligned} \widetilde{\Delta}_0 : U(\mathfrak{g}^\sigma[[u]]) &\to U(\mathfrak{g}[[u]]) \otimes U(\mathfrak{g}^\sigma[[u]]) \,, \\ b^{(0)}_\alpha &\mapsto t^{(0)}_\alpha \otimes 1 + 1 \otimes b^{(0)}_\alpha \,, \\ b^{(1)}_\mu &\mapsto t^{(1)}_\mu \otimes 1 + 1 \otimes b^{(0)}_\mu \,. \end{aligned} \tag{2.114}$$

Here it is essential that we use the UEAs $U(\mathfrak{g}[[u]])$ and $U(\mathfrak{g}^\sigma[[u]])$ rather than the Lie algebras themselves. This is because the tensor product of Lie algebras does not inherit a canonical Lie algebra structure, but the tensor product of associative algebras is an associative algebra. Note that the representation theory of a Lie algebra and its UEA are isomorphic. The representation $\rho_{V\otimes W}$ is then the composition of $\rho_V \otimes \rho_W$ with $\widetilde{\Delta}_0$.

The quantum correction to bulk-boundary OPE deforms this coideal structure to

$$\begin{aligned}\widetilde{\Delta}_\hbar : b_\alpha^{(0)} &\mapsto t_\alpha^{(0)} \otimes 1 + 1 \otimes b_\alpha^{(0)} \,, \\ b_\mu^{(1)} &\mapsto \left(t_\mu^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\nu} \{ t_\beta^{(0)}, t_\nu^{(0)} \} \right) \otimes 1 + 1 \otimes b_\mu^{(0)} - \hbar f_\mu{}^{\nu\alpha} t_\nu^{(0)} \otimes b_\alpha^{(0)} \,.\end{aligned} \tag{2.115}$$

This is analogous to the deformation of the coproduct on $U(\mathfrak{g}[[u]])$. We reviewed in section 1.3.4 how the deformation of the coproduct necessitates a deformation of $U(\mathfrak{g}[[u]])$ as an associative algebra, deforming it to the Yangian $\mathcal{Y}(\mathfrak{g})$ [42, 50]. In the same way, the deformation of the coideal structure, together with the deformation of $U(\mathfrak{g}[[u]])$ to $\mathcal{Y}(\mathfrak{g})$, necessitates a deformation of $U(\mathfrak{g}^\sigma[[u]])$ as an associative algebra. We denote this deformed coideal subalgebra of the Yangian $\mathcal{B}(\mathfrak{g}, \mathfrak{h})$.

To determine $\mathcal{B}(\mathfrak{g}, \mathfrak{h})$, suppose we bring a bulk line operator supported at $z = 0$ to L in parallel in the absence of a boundary line operator. Then the calculation in section 2.3.1 shows that we obtain a boundary line operator with

$$b_{\alpha,V}^{(0)} = t_{\alpha,V}^{(0)} \,, \qquad b_{\mu,V}^{(1)} = t_{\mu,V}^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\nu} \{ t_{\beta,V}^{(0)}, t_{\nu,V}^{(0)} \} \tag{2.116}$$

This operation determines a map

$$\varphi : \mathcal{B}(\mathfrak{g}, \mathfrak{h}) \hookrightarrow \mathcal{Y}(\mathfrak{g}) \,, \qquad b_\alpha^{(0)} \mapsto t_\alpha^{(0)} \,, \qquad b_\mu^{(1)} \mapsto t_\mu^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\nu} \{ t_\beta^{(0)}, t_\nu^{(0)} \} \tag{2.117}$$

which coincides with the embedding of the twisted Yangian into the Yangian [61, 113, 24]. This shows that $\mathcal{B}(\mathfrak{g}, \mathfrak{h})$ is isomorphic to the twisted Yangian.

It is then possible to check the coideal structure explicitly by computing

$$\begin{aligned}\Delta_\hbar\Big(\varphi(b_\mu^{(1)})\Big) &= \Delta_\hbar \left(t_\mu^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\xi} \{ t_\beta^{(0)}, t_\xi^{(0)} \} \right) \\ &= \left(t_\mu^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\xi} \{ t_\beta^{(0)}, t_\xi^{(0)} \} \right) \otimes 1 + 1 \otimes \left(t_\mu^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\xi} \{ t_\beta^{(0)}, t_\xi^{(0)} \} \right) - \hbar f_\mu{}^{\xi\beta} t_\xi^{(0)} \otimes t_\beta^{(0)} \\ &= \left(t_\mu^{(1)} + \frac{\hbar}{4} f_\mu{}^{\beta\xi} \{ t_\beta^{(0)}, t_\xi^{(0)} \} \right) \otimes 1 + 1 \otimes \varphi(b_\mu^{(1)}) - \hbar f_\mu{}^{\xi\beta} t_\xi^{(0)} \otimes \varphi(b_\beta^{(0)}) \,.\end{aligned} \tag{2.118}$$

This coincides with the coideal structure in equation (2.115). Note that the final term in the second line prevents us from interpreting this as a coproduct on the twisted Yangian.

It is also possible to give an intrinsic definition of $\mathcal{B}(\mathfrak{g},\mathfrak{h})$ without realising it as a subset of $\mathcal{Y}(\mathfrak{g})$ [24].[7] It is generated as an associative algebra by $b^{(0)}_\alpha$ and $b^{(1)}_\mu$, which are required to obey

$$\begin{aligned}
&[b^{(0)}_\alpha, b^{(0)}_\beta] = f_{\alpha\beta}{}^\gamma b^{(0)}_\gamma\,, \qquad [b^{(0)}_\alpha, b^{(1)}_\nu] = f_{\alpha\nu}{}^\xi b^{(1)}_\xi\,,\\
&X_{\mu\nu}{}^{\rho\sigma}\Big([b^{(1)}_\rho, b^{(1)}_\sigma] - \hbar^2 S_{\rho\sigma}\Big(b^{(0)}_\bullet\Big)\Big) = 0\,,\\
&\Psi_{\lambda\mu\nu}{}^{\pi\rho\sigma}\Big([[b^{(1)}_\pi, b^{(1)}_\rho], b^{(1)}_\sigma] - \hbar^2 T_{\pi\rho\sigma}\Big(b^{(0)}_\bullet, b^{(1)}_\bullet\Big)\Big) = 0\,.
\end{aligned} \tag{2.119}$$

For

$$X_{\mu\nu}{}^{\rho\sigma} = \delta_\mu{}^\rho\delta_\nu{}^\sigma + \sum_\alpha \bar{c}_\alpha^{-1} f_{\mu\nu}{}^\alpha f_\alpha{}^{\rho\sigma}\,, \qquad \Psi_{\mu\nu\lambda}{}^{\pi\rho\sigma} = \delta_\mu{}^\pi\delta_\nu{}^\rho\delta_\nu{}^\sigma + 2c_{\mathfrak{g}}^{-1} f_{\mu\nu}{}^\alpha f_{\lambda\alpha}{}^\pi c^{\rho\sigma} \tag{2.120}$$

projection operators and S, T given by

$$\begin{aligned}
S_{\rho\sigma}\Big(b^{(0)}_\bullet\Big) &= \frac{1}{18} f^{\alpha\mu}{}_\rho f^{\beta\nu}{}_\sigma f^\gamma{}_{\mu\nu}\{b^{(0)}_\alpha, b^{(0)}_\beta, b^{(0)}_\gamma\}\,,\\
T_{\pi\rho\sigma}\Big(b^{(0)}_\bullet, b^{(1)}_\bullet\Big) &= \frac{1}{24}\Big(f_{\mu\nu}{}^\gamma f_\pi{}^{\alpha\mu} f_\rho{}^{\beta\nu} f_{\gamma\sigma}{}^\xi + f_{\pi\rho}{}^\gamma f_\gamma{}^{\alpha\delta} f_\sigma{}^{\beta\mu} f_{\delta\mu}{}^\lambda\Big)\{b^{(0)}_\alpha, b^{(0)}_\beta, b^{(1)}_\lambda\}\,.
\end{aligned} \tag{2.121}$$

(Here we have employed notation introduced in appendix B.1.) Following the results of [50] for $\mathcal{Y}(\mathfrak{g})$ we expect the $\mathcal{O}(\hbar^2)$ and $\mathcal{O}(\hbar^3)$ deformations to the twisted half-loop algebra described by S and T arise from 2-loop anomalies on the boundary Wilson line, though do not attempt to prove this here. We emphasise that the above relations follow as a necessary consequence of the embedding $\mathcal{B}(\mathfrak{g},\mathfrak{h}) \hookrightarrow \mathcal{Y}(\mathfrak{g})$ deduced from the bulk-boundary OPE.

2.3.3 Framing anomaly on the orbifold

We saw in subsection 1.3.4 that line operators in CS_4 suffer from an anomaly if they curve in Σ. This can be compensated by allowing the spectral parameter to vary continuously as the line operator bends so that

$$z - \frac{\hbar \mathbf{h}^\vee}{2\pi}\varphi \tag{2.122}$$

is constant. Here φ is the angle between the Wilson line and some framing of Σ.

[7] The cases $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{h}$ are exceptional. The required deformations in these cases can be found in [23, 24]

Because the framing anomaly is local to the line operator it is unchanged for bulk line operators on the orbifold. One might worry that when a bulk line operator crosses the apparent boundary $\pi = \{x = 0\} \subset \mathcal{V}$ that it may self-interact with the 'image' part of the propagator and so there may be extra contributions to the anomaly. Fortunately these are finite and do not affect the result. One important consequence of this is that the incoming and outgoing line operators in the configuration giving rise to a K-matrix must have the same angle of incidence and reflection, i.e., the corresponding line operator on the orbifold must be smooth. If this condition is not satisfied then one can easily verify that the classical ℓ-matrix depends on the difference between these angles, and so diffeomorphism invariance in the Σ direction is broken.

Boundary line operators also suffer from the framing anomaly, but we cannot compensate the curving of a boundary Wilson line by changing the spectral parameter. This issue can be overcome by choosing our framing so that it respects the $\mathbb{Z}_2$ action on Σ by $\mathcal{R}$. This is equivalent to a framing of $\overline{\Sigma}$ which is tangent to the boundary. Not all manifolds with boundary admit such a framing, for example, this excludes the disc $\overline{\Sigma} = D^2$. Fortunately $\overline{\Sigma} = \mathbb{R} \times \mathbb{R}_{\leq 0}$ and $S^1 \times \mathbb{R}_{\leq 0}$ are permitted, which are the most relevant for the study of quantum integrable models with boundary.

Note that the action of the reflection $\mathcal{R}$ on $\mathcal{V}$ by itself is actually anomalous, since it is not consistent with the framing anomaly. In particular, the mirror image of a line operator will curve in the opposite direction, necessitating a shift in the spectral parameter with the opposing sign. Similarly, swapping the sign of z by itself is not a symmetry. This is bourne out when considering examples of R-matrices, for example, Yang's rational R-matrix does not obey

$$R_{12}(z) = R_{12}(-z)\,. \tag{2.123}$$

Only the combined action, $\mathcal{P}$, is a symmetry of the quantum theory.

As in the bulk, the framing anomaly means that the VEV of the configuration appearing in figure 2.4 does not actually give a solution to the BYBE. This is because, as depicted, the bulk line operator reflects off the boundary with a non-vanishing angle of incidence. By the framing anomaly, changing this angle is equivalent to shifting the spectral parameter at order $\hbar$, which of course modifies the K-matrix. If we consider the diagrammatic representation of the BYBE in figure 2.5 it's clear that two reflecting Wilson lines have different angles of incidence, and so a distinct K-matrix is generated by each.

We know from the discussion in subsection 1.3.4 how to resolve this issue. To obtain a true solution to the BYBE we should instead compute the VEV of the following configuration.

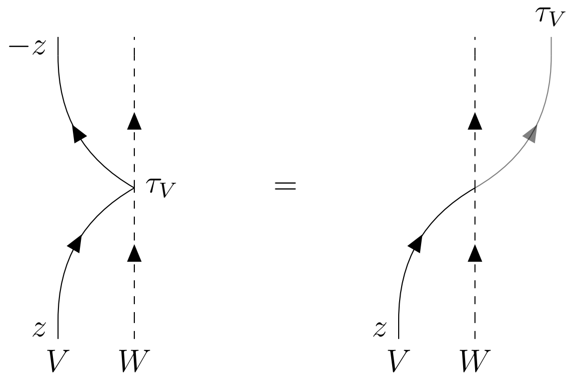


Fig. 2.11 Corrected K-matrix in CS_4

Since the incoming and outgoing Wilson lines are parallel, it's possible to concatenate multiple copies of this and to join it to the corrected representation of an R-matrix given in figure 1.10.

If the angle of incidence in figure 2.4 is θ, then the correction from the framing anomaly means it will actually compute

$$K(z + \hbar \mathbf{h}^{\vee}(\pi - 2\theta)/4\pi)\,. \tag{2.124}$$

As in the bulk, at order $\hbar$ this shift is irrelevant and the calculation in section 2.2.4 is still reliable.

We continue not to include the necessary curving of Wilson lines in most of our diagrams. It is understood that whenever we compute the VEV of a configuration where line operators tend to infinity that they eventually curve to the vertical.

2.3.4 RTT presentation

In subsection 2.3.2 we argued that a family of line operators supported on the singular line L are labelled by representations of the twisted Yangian, $\mathcal{B}(\mathfrak{g},\mathfrak{h})$. This was achieved by recognising that the embedding $\mathcal{B}(\mathfrak{g},\mathfrak{h}) \hookrightarrow \mathcal{Y}(\mathfrak{g})$ could be realised in CS_4 by bringing a bulk line operator supported at $z = 0$ to the boundary in parallel. In this section we show how the RTT presentation of $\mathcal{B}(\mathfrak{g},\mathfrak{h})$ can be obtained from CS_4. This provides a second demonstration that boundary line operators are labelled by representations of the twisted Yangian.

Much like in the bulk, the idea is to use the rational R-matrix associated to two copies of the defining vector representation of $\mathfrak{g}$ to constrain a generic boundary line operator. Consider a generalised boundary line operator

$$\mathcal{W}_W[L,A] = \mathrm{P}\exp\left(\int_L \sum_{n=0}^{\infty}\left(\frac{1}{2n!}b^{(2n)}_{\alpha,W}\iota_L^*\partial_z^{2n}A^\alpha + \frac{1}{(2n+1)!}b^{(2n+1)}_{\mu,W}\iota_L^*\partial_z^{2n+1}A^\mu\right)\right) \quad (2.125)$$

as defined in (2.39). For the moment we make no assumptions about the algebra of the $b^{(2n)}_{\alpha,W}$ and $b^{(2n+1)}_{\mu,W}$.

We can probe this operator with a bulk Wilson line in the representation U. In particular, using the bases $\{\langle i|\}_{i=1}^n$ and $\{|j\rangle\}_{j=1}^n$ of U and $U^\vee$ respectively, consider the VEV of the following configuration.

$B^i{}_{j,W}(z) \quad =$

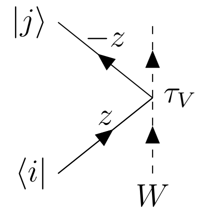


Fig. 2.12 Boundary transfer matrix

This correlator defines the matrix elements $B^i{}_{j,W}(z)$ of the boundary transfer matrix $B_W(z)$. Note that these take values in $\mathrm{End}\,W$. For the remainder of this section we suppress the W index on the boundary transfer matrix and its elements unless this is ambiguous.

Expanding $B(z)\tau_U^{-1}$ as a formal power series in $1/z$ allows us to extract the coefficient of $1/z^{m+1}$, which we denote by $\tilde{b}^i{}_j[m]$.

$$B^i{}_j(z) = \left(\delta^i{}_k + \hbar\sum_{n=0}^{\infty}\frac{\tilde{b}^i{}_k[m]}{z^{n+1}}\right)\tau^k{}_j\,. \quad (2.126)$$

We can view these coefficients as defining the boundary line operator $\mathcal{W}_W[L,A]$, instead of the parameters $b^{(2n)}_\alpha$ and $b^{(2n+1)}_\mu$ appearing in (2.125). A minor modification of the calculation in (2.2.4) shows that

$$\begin{aligned}
\tilde{b}^i{}_j[0] &= \frac{\hbar}{4}c^{ab}\langle i|t_{a,U}\sigma_U(t_{a,U})|j\rangle + 2\hbar c^{\alpha\beta}\langle i|t_{\alpha,U}|j\rangle b^{(0)}_\beta \\
\tilde{b}^i{}_j[2m] &= 2\hbar c^{\alpha\beta}\langle i|t_{\alpha,U}|j\rangle b^{(2m)}_\beta + \mathcal{O}(\hbar^{2m+1}) \quad \text{for} \quad m \geq 1 \\
\tilde{b}^i{}_j[2m+1] &= -2\hbar c^{\mu\nu}\langle i|t_{\mu,U}|j\rangle b^{(2m+1)}_\nu + \mathcal{O}(\hbar^{2m+2}) \quad \text{for} \quad m \geq 0\,,
\end{aligned} \quad (2.127)$$

where the higher order terms on the right hand side are determined by the Feynman diagram expansion.

We wish to find the algebra obeyed by the coefficients $\tilde{b}^i{}_j[m]$. This is achieved by constraining the boundary transfer matrix using identities relating correlators. The most important relations follows from the BYBE

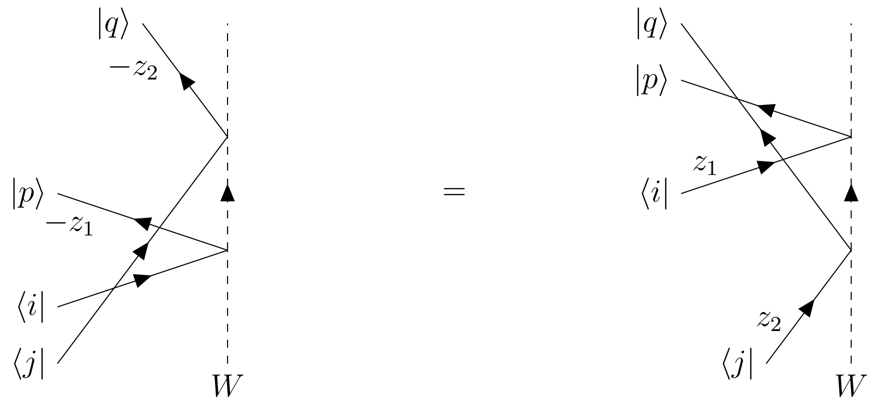


Fig. 2.13 Quaternary relations

where the lines supported at $z_1, z_2 \in \mathbb{C}$ are in the representation U. Algebraically, these conditions are

$$\begin{aligned} &R^{ij}{}_{k\ell}(z_1 - z_2)B^k{}_m(z_1)R^{\ell m}{}_{np}(z_1 + z_2)B^n{}_q(z_2) \\ &\quad = B^j{}_\ell(z_2)R^{i\ell}{}_{kn}(z_1 + z_2)B^k{}_m(z_1)R^{nm}{}_{qp}(z_1 - z_2) \end{aligned} \tag{2.128}$$

or in short

$$R_{12}(z_1-z_2)B_1(z_1)R_{21}(z_1+z_2)B_2(z_2) = B_2(z_2)R_{12}(z_1+z_2)B_1(z_1)R_{21}(z_1+z_2)\,. \tag{2.129}$$

The subscripts indicate on which factor of U the operators act. These are sometimes referred to as the quaternary relations.

Even for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ the quaternary relation alone is not sufficient to ensure that the algebra obeyed by the coefficients $\tilde{b}^i{}_j[m]$ is isomorphic to $\mathcal{B}(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_{n-k}(\mathbb{C}) \oplus \mathfrak{gl}_k(\mathbb{C}))$. The algebra has too many central elements.

To eliminate some of these note the equivalence of configurations illustrated in figure 2.14. This holds by construction in CS_4, and in fact is essentially the statement that the choice of apparent boundary π is not canonical. The dots in the right hand diagrams indicate an insertion of τ_U^2 on the Wilson line. Given that $\tau \in \mathrm{SL}_n(\mathbb{C})$ we know that $\tau_U^2 = \exp(2\pi i k/n)\mathbb{1}_U$ for some $k \in \mathbb{Z}$. Therefore, taking the VEV leads to the constraint

$$B^i{}_k(z)B^k{}_j(-z) = \exp(2\pi i k/n)\delta^i{}_j\,, \tag{2.130}$$

or more succinctly $B(z)B(-z) \propto \mathbb{1}_U$. This is known as boundary unitarity.

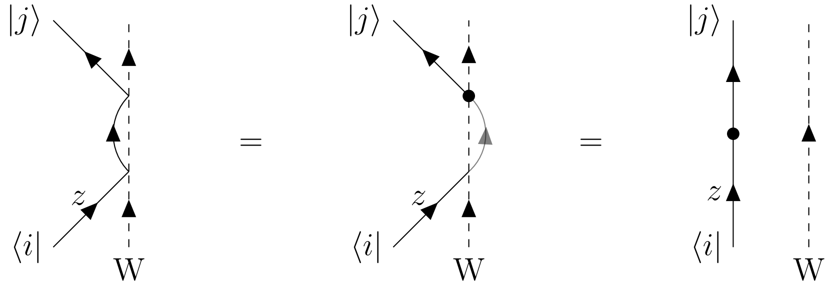


Fig. 2.14 Boundary unitarity

A similar relation was found in the bulk which we termed bulk unitarity 1.3.4. It imposed no constraints on the transfer matrix $T(z)$, instead giving a diagrammatic representation to $T^{-1}(z)$. In contrast, boundary unitarity is essential in defining the twisted Yangian in the RTT presentation.

Expanding out the quaternary relation and boundary unitarity gives relations between the coefficients $\tilde{b}^i{}_j[m]$. These imply that they furnish a representation of the twisted Yangian $\mathcal{B}(\mathfrak{gl}_n(\mathbb{C}), \mathfrak{gl}_{n-k}(\mathbb{C}) \oplus \mathfrak{gl}_k(\mathbb{C}))$ [127]. Note, however, the coefficients do not obey the algebra introduced in subsection 2.3.2. Indeed, these two presentations are related by the Feynman diagram expansion which at leading order in $\hbar$ is given in equation (2.127).

To obtain twisted Yangians for simple $\mathfrak{g}$ we need further constraints. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the final condition we require is a restriction on the Sklyanin determinant. This is the boundary analogue of the quantum determinant in the bulk, and it involves both $B(z)$ and the R-matrix. It can be expressed as

$$\begin{aligned} &\varepsilon B_1(z)R_{21}(2z+\hbar)R_{31}(2z+2\hbar)\cdots R_{n1}(2z+(n-1)\hbar)\, B_2(z+\hbar)R_{32}(2z+3\hbar) \\ &\qquad \cdots R_{n2}(z+n\hbar)B_3(z+2\hbar)\cdots B_n(z+(n-1)\hbar)(z) = \varepsilon\mathbb{1}\,, \end{aligned} \tag{2.131}$$

where the numerical suffixes on B and R indicate the particular copy of U in the tensor product $U^{\otimes n}$ on which they act. The totally antisymmetric ε tensor on the left contracts with the free index on B_1, together with the free indices on the R_{j1} appearing immediately to the right of it. The Sklyanin determinant itself, $\mathrm{sdet}(B)(z)$, is obtained by evaluating the left hand side of (2.131) on the trivial permutation. Then (2.131) is $\mathrm{sdet}(B)(z) = 1$. This condition removes the remaining central elements from the algebra defined by the quaternary relations and boundary unitarity [127].[8]

[8] In this paper a different normalization is used for the Sklyanin determinant. This is to ensure compatibility with a particular choice of embedding of the twisted Yangian into the Yangian, which

The constraint on the Sklyanin determinant can be realised in CS_4 using the vertex corresponding to ε discussed in subsection 1.3.4. The required sequence of diagrams is

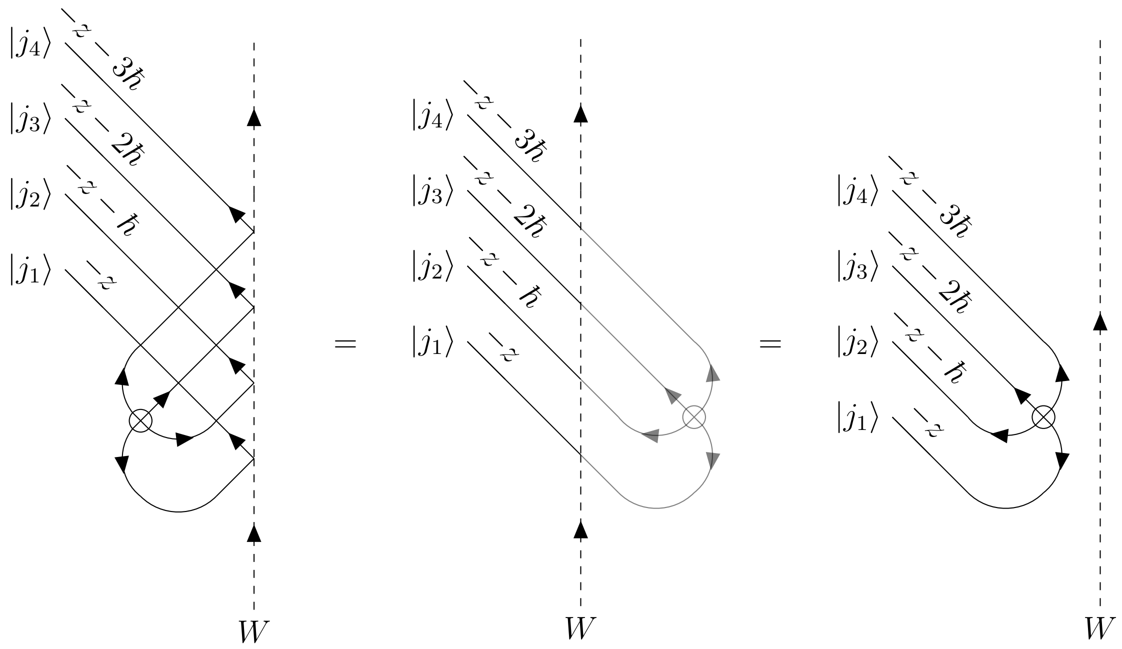


Fig. 2.15 Sklyanin determinant

illustrated here for the case $\mathfrak{g} = \mathfrak{sl}_4$. The spectral parameter at the vertex takes the value $z+2\hbar$ in the left hand diagram, and $-z-2\hbar$ in the two subsequent diagrams. The framing anomaly then shifts the spectral parameters of the Wilson lines as they curve to tend to infinity in parallel. This exactly reproduces the arguments of the boundary transfer matrix and R-matrix appearing in the Sklyanin determinant (2.131). Note that in going from the first picture to the second we've used the fact that $\det \tau_U = 1$.

We emphasise that to obtain both boundary unitarity and the constraint on the Sklyanin determinant it is essential to work on the orbifold, since in going from the second picture to the third we've acted with a diffeomorphism of $\widetilde{\mathcal{V}}$ which does not fix π. Once again, the virtue of CS_4 is that the condition $\mathrm{sdet}(B) = 1$ is obeyed by construction.

This provides a second demonstration that the family of boundary line operators defined in (2.125) are labelled by representations of twisted Yangians.

Remark. The boundary transfer matrix is nothing more than a K-matrix with boundary degrees of freedom for which the reflecting bulk Wilson line is in the representation

from the point of view of gauge theory is unnatural. This discrepancy does not affect the applicability of their results in this context.

U. In the same way that vertices built from invariants in tensor powers of U can be used to constrain the boundary transfer matrix, a generic K-matrix associated to representations V and W of $\mathfrak{g}$ and $\mathfrak{h}$ must obey identities arising from vertices involving V. The simplest example of such a constraint arises for $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_{n-k}(\mathbb{C}) \oplus \mathfrak{sl}_k(\mathbb{C}) \oplus \mathbb{C})$, $V = U$, $W \cong \mathbb{C}$, leading to the standard K-matrix in example 2.2.1 solving the BYBE for Yang's rational R-matrix. This K-matrix obeys the Sklyanin determinant condition. In section 2.3.5 we will show that for a given classical k-matrix, to all orders in $\hbar$ there is a unique quasi-classical K-matrix that obeys the BYBE and has unit Sklyanin determinant.

Now let's apply this construction to the remaining simple Lie algebras. Boundary unitarity and the quaternary relation are unchanged, so we just need to identify appropriate analogues of the Sklyanin determinant.

The fundamental representation of $\mathfrak{so}_n(\mathbb{C})$ admits a symmetric invariant tensor, $\delta \in \mathrm{S}^2 U^\vee$, and the fundamental representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ admits an antisymmetric invariant tensor $\omega \in \wedge^2 U^\vee$. Since these vertices are totally (anti)symmetric, they quantize without anomalies provided the two incoming Wilson lines leave the vertex directly opposite to one another. We will deal with both cases simultaneously, and refer to the invariant tensor as η.

Guided by the construction of the Sklyanin determinant in terms of line operators, we consider the configurations

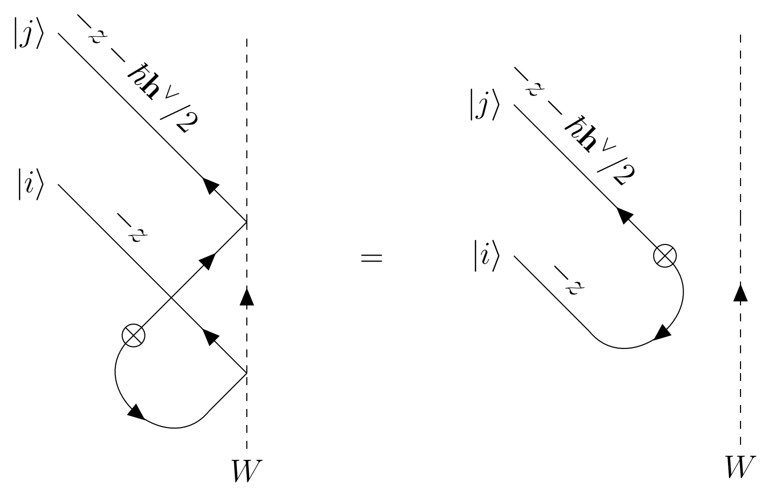


Fig. 2.16 Symmetry relations

from which we deduce that

$$\eta_{mn} B^m{}_k(z)\, R^{nk}{}_{\ell i}(2z + \hbar \mathbf{h}^\vee/2)\, B^\ell{}_j(z + \hbar \mathbf{h}^\vee/2) = \eta_{ij}\,. \tag{2.132}$$

In this form, this relation closely resembles the Sklyanin determinant condition for $\mathfrak{sl}_n(\mathbb{C})$. We believe it is equivalent to the symmetry relations described in [84] for the twisted Yangians corresponding to $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$ in the RTT presentation. (Indeed, we argue momentarily that this must be the case.) The condition (2.132) is natural from the point of view of the gauge theory.

We now briefly turn to the exceptional Lie algebras. As reviewed in section 1.3.4, in [51] RTT presentations of the Yangian for all of the exceptional Lie algebras apart from $\mathfrak{e}_8$ are constructed using CS_4. This is achieved using vertices associated to invariants in tensor powers of the fundamental representation U with smallest dimension. By deriving analogues of the Sklynanin determinant for these vertices, it is straightforward to define twisted Yangians for all of the exceptional Lie algebras apart from $\mathfrak{e}_8$.

We have seen in sections 2.3.1 & 2.3.2 how the OPE between a bulk and boundary line operator gives rise to coideal structure of the twisted Yangian. Furthermore, we saw that taking the boundary line operator to be trivial gives the inclusion $\mathcal{B}(\mathfrak{g},\mathfrak{h}) \hookrightarrow \mathcal{Y}(\mathfrak{g})$. By probing this with a reflecting Wilson line we obtain the coideal structure and inclusion in the RTT presentation. The relevant diagrams are illustrated below.

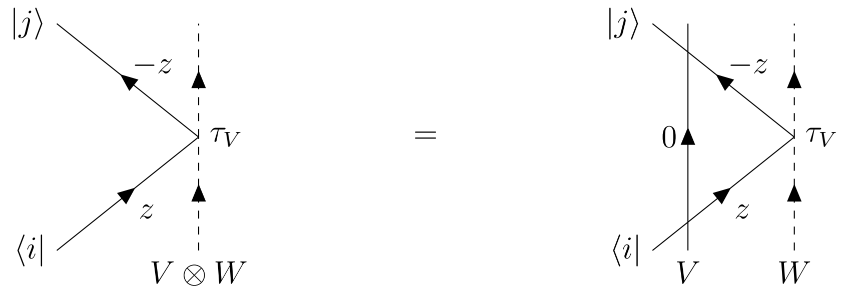


Fig. 2.17 Coideal structure in the RTT presentation

Taking their VEVs gives

$$B^i{}_{j,V\otimes W}(z) = (T^{-1})^i{}_{k,V}(-z) T^\ell{}_{j,V}(z) \otimes B^k{}_{\ell,W}(z)\,, \tag{2.133}$$

where we've made the representations explicit, or more abstractly

$$\widetilde{\Delta}_\hbar(B^i{}_j(z)) = (T^{-1})^i{}_k(-z) T^\ell{}_j(z) \otimes B^k{}_\ell(z)\,. \tag{2.134}$$

(Here we are not merely suppressing representation index, but viewing $B^i{}_j(z)$ and $T^i{}_j(z)$ as taking values in $\mathcal{B}(\mathfrak{g},\mathfrak{h})$ and $\mathcal{Y}(\mathfrak{g})$ respectively.) Note that we've used bulk unitarity to identify the object generated at the second crossing in the diagram on the right hand side of figure 2.17 as $T^{-1}(z)$.

If we take the boundary Wilson line to be trivial then we obtain an embedding of the twisted Yangian as a subalgebra the Yangian. We have

$$B^i{}_{j,V}(z) = (T^{-1})^i{}_{k,V}(-z)T^\ell{}_{j,V}(z)K^k{}_\ell(z) \tag{2.135}$$

On the right hand side appears the quasi-classical K-matrix without boundary degrees of freedom generated by a bulk reflecting Wilson line in the fundamental representation U. It was explicitly constructed up to scale in subsection 2.2.6 for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ and all σ inner.

When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ the relation (2.134) is exactly the formula for the coproduct appearing in [127]. It slightly differs from the formula in [84] for the remaining classical Lie algebras, but this discrepancy can be accounted for by writing $T^{-1}(z)$ in terms of the transfer matrix for the representation dual to V using the framing anomaly. This shows that the symmetry relations (2.132) do indeed define twisted Yangians isomorphic to those appearing in [84].

Remark. There exist further manipulations of diagrams which reproduce identities appearing in the literature. For example, it's easy to identify diagrams leading both to K-matrix fusion and the boundary transfer matrix of the dual representation [82].

2.3.5 Uniqueness of the rational K-matrix

Let $\sigma = \operatorname{conj}\tau$ be an involutive inner automorphism of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. We have seen in this chapter that the VEV of a reflecting bulk Wilson line in the fundamental representation U of $\mathfrak{g}$, perhaps in the presence of a boundary Wilson line in a 1 dimensional representation of $\mathfrak{h}$, generates a rational K-matrix without boundary degrees of freedom with the following properties.

- It obeys the BYBE for Yang's rational R-matrix, given in example 1.3.1.
- It has an expansion

$$K(z) = \tau_U + \hbar\ell_{U\otimes W}(z)\tau_U + \mathcal{O}(\hbar^2) \tag{2.136}$$

  for $\ell_{U\otimes W}(z)$ the image of the classical ℓ-matrix from equation (2.61) in the tensor product of the fundamental U with a 1 dimensional representation W.
- It has unit Sklyanin determinant.
- It's invariant under simultaneous rescalings of $\hbar$ and z.

In this section we prove that this uniquely determines the K-matrix to all orders in $\hbar$. The proof follows the same argument employed in [51] to prove the uniqueness of the rational R-matrix. Afterwards, we discuss the generalisation to arbitrary irreducible representations V of simple $\mathfrak{g}$. (These are of course required to lift to $\mathcal{Y}(\mathfrak{g})$.)

A result of this kind has not appeared in the literature previously, largely because the notion of a quasi-classical K-matrix was first introduced by the author in [29]. It is important in extending results which hold to first order in $\hbar$ to all orders. We used it repeatedly in subsection 2.2.6.

Proof. We start by assuming that we have two rational K-matrices with the properties listed above which agree up to order $\hbar^{r-1}$ for some $r \geq 2$. In particular we assume that they have the same classical limit τ_U. The two K-matrices are hence related by

$$K'(z) = K(z) + \frac{\hbar^r}{z^r}\delta\ell_r\tau_U + \mathcal{O}(\hbar^{r+1}) \in \operatorname{End} U\,, \tag{2.137}$$

for some $\delta\ell_r \in \operatorname{End} U$. The proof proceeds in two steps. We begin by comparing the Sklyanin determinant of the two K-matrices, finding that

$$\begin{aligned}\operatorname{sdet}(K')(z) &= \operatorname{sdet}(K)(z)\\ &+ \varepsilon_{i_1\dots i_n}\frac{\hbar^r}{z^r}\sum_{j=1}^{n}(\tau_U)^{i_1}{}_1(\tau_U)^{i_2}{}_2\dots(\tau_U)^{i_{j-1}}{}_{j-1}(\delta\ell_r)^{i_j}{}_k(\tau_U)^k{}_j(\tau_U)^{i_{j+1}}{}_{j+1}\dots(\tau_U)^{i_n}{}_n + \mathcal{O}(\hbar^{r+1})\,.\end{aligned} \tag{2.138}$$

Here we've used the fact that at lowest order in $\hbar$, the R-matrix is equal to $\mathbb{1}_{U\otimes U}$ and the K-matrix is equal to τ_U. From this we can deduce that

$$\begin{aligned}\varepsilon_{i_1\dots i_n}\sum_{j=1}^{n}(\tau_U)^{i_1}{}_1(\tau_U)^{i_2}{}_2\dots(\tau_U)^{i_{j-1}}{}_{j-1}(\delta\ell_r)^{i_j}{}_k(\tau_U)^k{}_j(\tau_U)^{i_{j+1}}{}_{j+1}\dots(\tau_U)^{i_n}{}_n &= \sum_{j=1}^{n}(\tau_U^{-1}\delta\ell_r\tau_U)^{i_j}{}_j\\ = \operatorname{tr}(\tau_U^{-1}\delta\ell_r\tau_U) = \operatorname{tr}(\delta\ell_r) = 0 &\end{aligned} \tag{2.139}$$

where the first equality uses the fact that τ_U has determinant 1. This shows that $\delta\ell_r$ lies in $\mathfrak{sl}_n(\mathbb{C})$ viewed as a subspace of $\operatorname{End} U$.

The second step is to substitute equation (2.137) into the BYBE, giving

$$\begin{aligned}&K_2(v)R_{12}(u+v)K_1(u)R_{21}(u-v) + \hbar^r\left(\frac{\delta\ell_{r,2}}{v^r} + \frac{\delta\ell_{r,1}}{u^r}\right)\tau_{U,1}\tau_{U,2} + \mathcal{O}(\hbar^{r+1})\\ &\quad = R_{12}(u-v)K_1(u)R_{21}(u+v)K_2(v) + \hbar^r\left(\frac{\delta\ell_{r,2}}{v^r} + \frac{\delta\ell_{r,1}}{u^r}\right)\tau_{U,1}\tau_{U,2} + \mathcal{O}(\hbar^{r+1})\end{aligned} \tag{2.140}$$

at lowest non-trivial order in $\hbar$. The above holds identically, so in order to get a non-trivial constraint we expand to order $\hbar^{r+1}$. Note that the term of order $\hbar^{r+1}$ in the difference between the two K-matrices will not contribute for precisely the same reason that term involving $\delta\ell_r$ didn't give a constraint at order $\hbar^r$. Hence, the only non-trivial terms come from expanding the R-matrices to first order in $\hbar$. Recall from section 1.3.3 that

$$R(z) = \mathbb{1}_{U\otimes U} + \frac{\hbar}{z}C + \mathcal{O}(\hbar^2)\,, \tag{2.141}$$

where we have introduced $C = c_{U\otimes U} = c^{ab}t_{a,U}\otimes t_{b,U}$. $C_{12} = C_{21} = C$. Expanding to order $\hbar^{r+1}$ gives

$$\begin{aligned}\hbar^{r+1}&\left(\frac{\delta\ell_{r,2}\sigma_{U,2}(C)}{v^r(u+v)} + \frac{\delta\ell_{r,2}\sigma_{U,1}\sigma_{U,2}(C)}{v^r(u-v)} + \frac{\sigma_{U,2}(C)\delta\ell_{r,1}}{(u+v)u^r} + \frac{\delta\ell_{r,1}\sigma_{U,1}\sigma_{U,2}(C)}{(u-v)u^r}\right) + \mathcal{O}(h^{r+2})\\ &= \hbar^{r+1}\left(\frac{\sigma_{U,1}(C)\delta\ell_{r,2}}{v^r(u+v)} + \frac{C\delta\ell_{r,2}}{v^r(u-v)} + \frac{\delta\ell_{r,1}\sigma_{U,1}(C)}{(u+v)u^r} + \frac{C\delta\ell_{r,1}}{(u-v)u^r}\right) + \mathcal{O}(h^{r+2})\,,\end{aligned} \tag{2.142}$$

where we have defined $\sigma_U = \operatorname{conj}\tau_U$. For this to hold in general, it must hold at order $\hbar^{r+1}$. Furthermore, for $r \geq 2$ the functions

$$\frac{1}{u^r(u+v)}\,,\qquad \frac{1}{u^r(u-v)}\,,\qquad \frac{1}{v^r(u+v)}\,,\qquad \frac{1}{v^r(u-v)} \tag{2.143}$$

are linearly independent. This means we must have

$$\begin{aligned}\delta\ell_{r,2}\sigma_{U,2}(C) &= \sigma_{U,1}(C)\delta\ell_{r,2}\,, \qquad & \delta\ell_{r,2}\sigma_{U,1}\sigma_{2,U}(C) &= C\delta\ell_{r,2}\,,\\ \sigma_{U,2}(C)\delta\ell_{r,1} &= \delta\ell_{r,1}\sigma_{U,1}(C)\,, \qquad & \delta\ell_{r,1}\sigma_{1,U}\sigma_{2,U}(C) &= C\delta\ell_{r,1}\,.\end{aligned} \tag{2.144}$$

It turns out that all four of these relations are equivalent, so we'll concentrate on the last. Using $\sigma_{1,U}\sigma_{2,U}(C) = C$ it simplifies to

$$[C, \delta\ell_{r,1}] = c^{ab}[t_{a,U}, \delta\ell_r]\otimes t_{b,U} = 0\,. \tag{2.145}$$

Since the $t_{a,U}$ are linearly independent and c is non-degenerate, we must have

$$[\delta\ell_r, t_{a,U}] = 0 \tag{2.146}$$

for all a. It then follows from Schur's lemma that $\delta\ell_r \propto \mathbb{1}_U$. Since $\operatorname{tr}(\delta\ell_r) = 0$, we deduce that $\delta\ell_r = 0$.

We have therefore shown that if two K-matrices with the prescribed features agree to order $\hbar^r$ for $r \geq 2$, then they must also agree to order $\hbar^{r+1}$. Inducting on r gives uniqueness. □

Suppose that instead of taking the bulk Wilson line to be in the fundamental representation of $\mathfrak{sl}_n(\mathbb{C})$, we choose it to be in an arbitrary irreducible representation V of a simple $\mathfrak{g}$. (We, of course, require that this lifts to $\mathcal{Y}(\mathfrak{g})$.) There may no longer be suitable vertices which lead to analogues of the Sklyanin determinant, and so the first part of the proof fails. The second part goes thorough, however, with the only difference being that when invoking Schur's lemma at the end we may now only infer that $\delta\ell_r$ is proportional to $\mathbb{1}_V$. By rescaling $K'(z)$ by an appropriate $F(z,\hbar) = 1 + \mathcal{O}(\hbar^r)$ we can set $\delta\ell_r = 0$. Proceeding inductively we therefore learn that the K-matrix is unique up to an overall scale.

If we take V to be the fundamental representation U of any simple $\mathfrak{g}$ with the exception of $\mathfrak{e}_8$, then suitable vertices do exist. (This was touched upon in subsections 1.3.4 & 1.2.1, see [52] for further details.) Constraints derived from these vertices imply that $\delta\ell_r$ lies in the image of $\mathfrak{g}$ under the representation U. Together with the fact that $\delta\ell_r \propto \mathbb{1}_U$, this enough to set $\delta\ell_r = 0$.

Remark. It is also interesting to consider the case with boundary degrees of freedom, where K-matrices take values in $\mathrm{End}(V \otimes W)$ for V and W irreducible representations of $\mathfrak{g}$ and $\mathfrak{h}$. These must lift to the corresponding Yangian and twisted Yangian respectively. Recall that a quasi-classical K-matrix is taken to be of the form

$$K(z) = \tau_V \otimes \mathbb{1}_W + \hbar\ell_{V\otimes W}(z)(\tau_V \otimes \mathbb{1}_W) + \mathcal{O}(\hbar^2)\,, \tag{2.147}$$

and we further assume that the rational ℓ-matrix coincides with the one obtained from CS_4 and appearing in equation (2.61).

$$\ell_{V\otimes W}(z) = \frac{\ell_1}{z} = \frac{1}{4z}c^{ab}t_{a,V}\sigma_V(t_{b,V}) \otimes \mathbb{1}_W + \frac{2}{z}c^{\alpha\beta}t_{\alpha,V} \otimes b_{\beta,W}\,. \tag{2.148}$$

Following the second part of the above proof we find that $\delta\ell_r \in \mathrm{End}(V \otimes W)$ for $r \geq 2$ obeys a modification of equation (2.142) with the additional terms

$$\begin{aligned} &\cdots + \hbar^{r+1}\left(\frac{\delta\ell_{r,23}\ell_{1,13}}{v^r u} + \frac{\ell_{1,23}\delta\ell_{r,13}}{vu^r}\right) + \mathcal{O}(\hbar^{r+2}) \\ &= \cdots + \hbar^{r+1}\left(\frac{\delta\ell_{r,13}\ell_{1,23}}{u^r v} + \frac{\ell_{1,13}\delta\ell_{r,23}}{uv^r}\right) + \mathcal{O}(\hbar^{r+2})\,. \end{aligned} \tag{2.149}$$

These do not cancel trivially at order $\hbar^{r+1}$, since, for example, $\ell_{1,13}$ and $\delta\ell_{r,23}$ no longer commute in the $\operatorname{End} W$ factor. Fortunately for $r \geq 2$ the functions

$$\frac{1}{u^r v}\,, \qquad \frac{1}{uv^r} \tag{2.150}$$

are linearly independent from one another and those appearing in equation (2.143). The proof then continues as above and we find that $\delta\ell_r$ is $\mathfrak{g}$-invariant in the first factor. The constraints coming from the new terms in equation (2.149) imply that $\delta\ell_r$ is $\mathfrak{h}$-invariant in the second factor. Invoking Schur's lemma together these imply that $\delta\ell_r \propto \mathbb{1}_{V\otimes W}$, and so the K-matrix is unique up to scale.

It's also worth discussing why the proof (for K-matrices without boundary degrees of freedom) fails for $r = 1$. The second step goes wrong because the rational functions in (2.143) are no longer linearly independent. In particular

$$\frac{1}{u(u+v)} + \frac{1}{u(u-v)} = \frac{1}{v(u-v)} - \frac{1}{v(u+v)}\,. \tag{2.151}$$

This means that the constraints we can deduce are weaker than those for $r \geq 2$. They are given by

$$\begin{aligned} \delta\ell_2\sigma_{U,2}(C) + \delta\ell_2\sigma_{U,1}\sigma_{U,2}(C) &= \sigma_{U,1}(C)\delta\ell_2 + C\delta\ell_2\,, \\ \sigma_{U,2}(C)\delta\ell_1 + \delta\ell_2\sigma_{U,1}\sigma_{U,2}(C) &= \delta\ell_1\sigma_{U,1}(C) + C\delta\ell_2\,, \\ \delta\ell_1\sigma_{U,1}\sigma_{U,2}(C) + \delta\ell_2\sigma_{U,1}\sigma_{U,2}(C) &= C\delta\ell_1 + C\delta\ell_2\,. \end{aligned} \tag{2.152}$$

These can be simplified using $\sigma_{U,1}\sigma_{U,2}(C) = C$. Given that $\delta\ell_r$ lies in the image of $\mathfrak{sl}_n(\mathbb{C})$ under the representation U, the third equation holds trivially as a consequence of the $\mathfrak{g}$-invariance of c. The second is linearly dependent on the other two. Concentrating therefore on the first, we learn that

$$[\delta\ell_2, C + \sigma_2(C)] = c^{ab} t_{a,U} \otimes [\delta\ell, t_{b,U} + \sigma_U(t_{b,U})] = 0\,. \tag{2.153}$$

From the linear independence of the $t_{a,U}$ and the non-degeneracy of c we infer that

$$[\delta\ell, t_{a,U} + \sigma_U(t_{a,U})] = 0 \tag{2.154}$$

for all a. The map $Y \mapsto (Y + \sigma(Y))/2$ is the projection from $\mathfrak{sl}_n(\mathbb{C})$ onto $\mathfrak{h}$, and so we learn that $\delta\ell$ is $\mathfrak{h}$-invariant.

This result generalises to the fundamental U of more general simple $\mathfrak{g}$ with the exception of $\mathfrak{e}_8$. $\mathfrak{h}$-invariance of $\delta\ell$ is consistent with the quasi-classical K-matrices we obtained in subsection 2.2.6. In particular, when $\mathfrak{h}$ contains a copy of $\mathbb{C}$ they depend on a continuous parameter.

2.4 Discussion

In this chapter we have seen that almost all of the results of [50] can be carried over to the boundary case by studying CS_4 on a $\mathbb{Z}_2$ orbifold. There are a few differences which are worth emphasising, however.

The first is that unlike for R-matrices, prior to the work [29] there was no established notion of a quasi-classical K-matrix. Fortunately a suitable definition was not too difficult to deduce, and it has since been used elsewhere [83]. One downside of this is that there is no classification of solutions to the CBYBE, in stark contrast to its bulk counterpart which is very well understood. This was not an issue here, but we will see in chapter 3 that it causes problems for the trigonometric and elliptic cases.

The second is the importance of boundary Wilson lines in generating rational solutions to the BYBE, indeed, without them even the simplest family of solutions to the BYBE cannot be obtained in full generality. This is because the fixed point subalgebra of an involutive automorphism often contains an abelian summand.

The final difference is that we have not computed the anomaly to a boundary line operator leading to the Yangian deformation explicitly. This is not necessary to show that boundary Wilson lines are labelled by representations of twisted Yangians, even without resorting to the RTT presentation, since the coideal structure which arises from the bulk-boundary OPE determines the embedding of the twisted Yangian into the Yangian. Nonetheless, an intrinsic definition of the twisted Yanigan in the J-presentation has been given in [24], and it would be interesting to obtain this from CS_4. (We gave the defining relations in equation (2.119).) Unfortunately this is rather more complicated than the J-presentation of the Yangian. The second horrific relation is of order $\hbar^3$, and would appear to arise from a diagram involving 3 external legs. This may make the computation prohibitively difficult.

This work has a number of natural extensions which would be interesting to explore.

One simple possibility is to apply our construction to outer automorphisms σ. In fact much of this chapter goes through unmodified. The only major difference is that outer automorphisms need not fix representations. Consider, for example, $\sigma : \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$, $X \mapsto -X^{\mathrm{t}}$ corresponding to the symmetric pair $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$.

This automorphism exchanges the fundamental U with its dual $U^\vee$. In the diagrammatic representation of the BYBE this means that the arrow on a reflected fundamental Wilson line is reversed. The simplest analogue of a K-matrix is rather trivial: it takes values in $U^\vee \otimes U^\vee$, and so $\mathfrak{so}_n(\mathbb{C})$-invariance necessitates that it be proportional to the defining symmetric bilinear. It leads to soliton non-preserving boundary conditions in integrable models (see, e.g., [65]). The corresponding twisted Yangian was one of the first identified in [136], and notably admits an evaluation homomorphism so that all representations of $\mathfrak{so}_n(\mathbb{C})$ lift to it.

Recently Baxter Q-operators have been interpreted as 't Hooft lines in CS_4 [45]. It would be intriguing to study Q-operators at the boundary using our orbifold construction. See [77] for a discussion of Q-operators in the context of the open XXX spin chain.

Finally, it would be worthwhile to describe integrable boundary conditions for CIFTs along the lines of [53]. Given it is also expected that QIFTs can be described using CS_4, we speculate that this could lead to an understanding of the role of the BYBE in boundary scattering.

Chapter 3

Trigonometric and elliptic cases

Trigonometric and elliptic solutions of the BYBE play an important role in the study of quantum integrable models. The scattering matrices of the sine-Gordon model and affine Toda theory [82, 62] are trigonometric solutions to the BYBE, and boundary conditions for the 6 vertex model and XXZ spin chain together with its higher rank generalisations are determined by trigonometric K-matrices. Elliptic solutions to the BYBE play a less significant role in the context of QIFT, but determine the boundary conditions for the 8 vertex model and XYZ spin chain together with its higher rank generalisations.

The first examples of trigonometric and elliptic solutions to the BYBE date back to the works of Cherednik and Sklyanin [39, 152], where K-matrices with values in the fundamental of $\mathfrak{sl}_2(\mathbb{C})$ were introduced. Since then there has been a considerable body of work devoted to identifying and classifying examples of elliptic and trigonometric K-matrices.

The most general K-matrix for the Jimbo trigonometric R-matrix associated to the fundamental of $\mathfrak{sl}_2(\mathbb{C})$ was first obtained in [57, 82]. Extending this classification to all classical Lie algebras proved a formidable task, K-matrices for the fundamental of $\mathfrak{sl}_n(\mathbb{C})$ were discovered in [1, 80, 107, 177], and for the remaining classical Lie algebras in [107, 108]. Eventually this led to the full classification of solutions [116], subject to a regularity assumption.

The quantum group structures underlying these solutions are less well understood. The standard simplest trigonometric K-matrix for the fundamental of $\mathfrak{sl}_2(\mathbb{C})$ is underpinned by the q-Onsager algebra [15, 17]. Related algebras known as q-twisted Yangians were introduced in [128, 37], but it was not until Kolb's work on quantum symmetric pairs that these could be unified into a consistent theory of coideal subalgebras of quantum affine algebras [99].

Elliptic K-matrices for the fundamental of $\mathfrak{sl}_2(\mathbb{C})$ have classified in [56, 92], and higher rank analogues have been identified in [92, 100, 72]. We are unaware of any work investigating the quantum group structures underlying these solutions. (The boundary analogue of Felder's elliptic quantum group, associated to the dynamical YBE, has been studied for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ in [73].)

In chapter 2 rational solutions of the BYBE were generated as the VEVs of configurations of Wilson lines in CS_4. In this chapter we extended this construction to elliptic and trigonometric solutions of the BYBE. It is largely based on the paper [30].

3.1 Generalities

We have seen in subsection 1.3.5 that in order for CS_4 to generate trigonometric and elliptic solutions to the YBE we must choose (C, ω) to be $(\mathbb{C}/2\pi i\mathbb{Z}, \mathrm{d}z) \cong (\mathbb{C}^*, \mathrm{d}u/u)$ and $(\mathrm{T}^2_\tau, \mathrm{d}z) = (\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \mathrm{d}z)$ respectively. In the trigonometric case this requires imposing non-trivial boundary conditions as $\mathrm{Re}\, z \to \pm\infty$, whereas in the elliptic case we must perturb around a non-trivial vacuum. Let us for the moment ignore these technical details.

To obtain trigonometric and elliptic K-matrices we study CS_4 on the orbifold $\widetilde{\mathcal{V}} = \mathcal{V}/\mathbb{Z}_2$. Almost the whole discussion in subsection 2.2.1 and much of that in subsection 2.2.2 goes through without modification.

In brief, the $\mathbb{Z}_2$ action is generated by $\mathcal{P} : \mathcal{V} \to \mathcal{V}$, the composition of $z \mapsto -z$ on C with the reflection $\mathcal{R}$ on Σ. Since CS_4 continues to be local in Σ, without loss generality we take $\Sigma = \mathbb{R}^2$ and choose coordinates (x, y) such that

$$\mathcal{P} : (x, y, z, \bar{z}) \mapsto (-x, y, -z, -\bar{z})\,. \tag{3.1}$$

We then lift this to an involutive automorphism of the adjoint bundle of the gauge theory in such a way that it preserves the bilinear tr on the fibres and fixes the vacuum. This automorphism determines a $\mathbb{Z}_2$ action on the partial connection A by pullback which preserves the action of CS_4, $S_{\mathcal{V}}[A]$. We can then consistently define the path integral of CS_4 on $\widetilde{\mathcal{V}}$ as an integral over $\mathbb{Z}_2$-invariant field configurations with integrand $\exp(-S_{\widetilde{\mathcal{V}}}[A]/\hbar)$ for $S_{\widetilde{\mathcal{V}}}[A] = S_{\mathcal{V}}[A]/|\mathbb{Z}_2|$.

The bulk R-matrix is generated as the VEV of crossing Wilson lines away from the boundary, and diffeomorphism invariance in the Σ direction ensures that this is unaffected by the orbifold structure. K-matrices are generated as the VEVs of Wilson lines crossing $\pi = \{x = 0\} \subset \mathcal{V}$, which by exploiting $\mathbb{Z}_2$-invariance can be interpreted as reflecting off it. To generate K-matrices with boundary degrees of freedom we introduce boundary Wilson lines along the singular loci of the orbifold. The relevant configuration is illustrated in figure 2.4. The YBE and BYBE are manifest.

The only major differences are the following:

- Unlike for $C = \mathbb{C}$, the set of solutions $\{z_*\}$ to $2z = 0$ will have more than one element. For $C = \mathbb{C}/2\pi i\mathbb{Z}$ there are two: $z_* = 0, \pi i$, and for $C = \mathrm{T}^2_\tau$ there are four: $z_* = 0, 1/2, \tau/2, (1+\tau)/2$. The orbifold $\widetilde{\mathcal{V}}$ therefore has two and four lines of singular points respectively. On each of these lines the gauge field pulls back to a connection 1-form with values in a subalgebra $\mathfrak{p}_* \subset \mathfrak{g}$, and we can introduce a boundary Wilson line in a representation W_* of $\mathfrak{p}_*$. (In this chapter we write $\mathfrak{p}$ for the fixed point subalgebra of an involutive automorphism, reserving $\mathfrak{h}$ for the Cartan.) It is important to note that the representations W_* need not be isomorphic for different z_*.

- In the rational case we lifted the action of $\mathcal{P}$ to the adjoint bundle of the gauge theory using an involutive automorphism $\sigma \in \operatorname{Aut}\mathfrak{g}$. This was possible because the adjoint bundle was trivial. In elliptic case the adjoint bundle is non-trivial and so we cannot lift in such a naive way. In the trigonometric case the adjoint bundle is trivial, but nonetheless we will identify interesting examples of K-matrices associated to non-trivial bundle morphisms.

These facts are related since if $\mathcal{P}$ lifts to a non-trivial automorphism of the adjoint bundle, then the distinguished subalgebras $\mathfrak{p}_*$ associated to different fixed points z_* need not be isomorphic.

We will explore the freedom in lifting $\mathcal{P}$ to the adjoint bundle in the trigonometric and elliptic cases in sections 3.2 & 3.3 respectively. Before doing so it's useful to derive a general expression for the classical ℓ-matrix which applies in both cases. This requires working in a simplified gauge.

3.1.1 Computing the classical ℓ-matrix

In chapters 1 and 2 we studied CS_4 in an analogue of Lorenz gauge

$$* \,\mathrm{d} * A = 0\,, \tag{3.2}$$

where $*$ is the Hodge star induced by the metric $g = \delta^2_\Sigma + \omega\bar{\omega}/2$. Unfortunately in the trigonometric and elliptic cases deriving the explicit propagator in this gauge is prohibitively difficult.

In [50] this issue was circumvented when deriving classical r-matrices by constraining the contribution of the Feynman diagram 1.6 using its formal properties. This is not so straightforward in the boundary case because considerably less symmetry exists, and a greater number of diagrams contribute.

To overcome this issue we will instead work in the holomorphic gauge

$$\bar{A}_C = 0 \tag{3.3}$$

introduced in [53].[1] To understand this condition it's best to revisit the bulk rational case, where we can view this as the $R_C \to \infty$ limit of Lorenz gauge (3.2) for the metric

$$\tilde{g} = \delta_\Sigma^2 + \frac{R_C^2}{2}\omega\bar{\omega} = \mathrm{d}x^2 + \mathrm{d}y^2 + \frac{R_C^2}{2}\,\mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{3.4}$$

From 1.3.3 the bulk propagator is

$$P_{xy}(v) = \frac{c}{\pi}\bar{z}F(v)\,, \qquad P_{y\bar{z}}(v) = \frac{c}{2\pi}xF(v)\,, \qquad P_{\bar{z}x}(v) = \frac{c}{2\pi}yF(v) \tag{3.5}$$

for

$$F(v) = \frac{R_C^2}{(x^2 + y^2 + R_C^2|z|^2)^2}\,. \tag{3.6}$$

Now $\lim_{R_C\to\infty} F(v) = \pi\delta(x)\delta(y)$, and so in the $R_C \to \infty$ limit the propagator tends to

$$P_{xy}(v) = \frac{c}{z}\delta(x)\delta(y)\,, \qquad P_{y\bar{z}}(v) = 0\,, \qquad P_{\bar{z}x}(v) = 0\,. \tag{3.7}$$

Since all parts of the propagator involving $A_{\bar{z}}$ vanish this gauge is not suitable for any calculation involving the cubic interaction vertex. Nonetheless, it is sufficient to determine the classical r- and ℓ-matrix.

The generalisation of the propagator (3.7) to the elliptic and trigonometric cases is straightforward. We note that the only non-vanishing component can be expressed as

$$P_{xy}(v) = r(z)\delta(x)\delta(y) \tag{3.8}$$

for $r(z)$ the rational r-matrix. We simply replace this with the appropriate trigonometric or elliptic r-matrix.

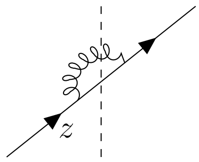
[1] It is important to note that this gauge is attainable for the choices of (C,ω) and boundary conditions leading to rational, trigonometric and elliptic solutions of the YBE. In contrast, it is certainly not attainable for the choice which led to the PCM in subsection 1.3.6.

To obtain the propagator on the orbifold $\widetilde{\mathcal{V}} = \mathcal{V}/\mathbb{Z}_2$ we employ the ansatz from subsection 2.2.3. Assuming that the lift of $\mathcal{P}$ to the adjoint bundle is determined by $\sigma \in \operatorname{Aut}\mathfrak{g}$, then the propagator is

$$\begin{aligned}\widetilde{P}_{xy}(v_1, v_2) &= P_{xy}(v_1 - v_2) + \sigma_2 P_{xy}(v_1 - \mathcal{P}v_2)\\ &= r(z_1 - z_2)\delta(x_1 - x_2)\delta(y_1 - y_2) + \sigma_2 r(z_1 + z_2)\delta(x_1 + x_2)\delta(y_1 - y_2)\end{aligned} \tag{3.9}$$

with all other components vanishing. This is easy to extend to less trivial lifts of $\mathcal{P}$ to the adjoint bundle, as will be discussed in subsection 3.2.2. Using this propagator we can obtain a general expression for the classical ℓ-matrix in terms of the classical r-matrix.

As in the rational case there are two types of diagrams which contribute to the classical ℓ-matrix. The first involves the self-interaction of a Wilson line on the orbifold as it crosses π.



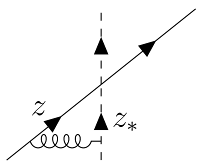
Using the propagator (3.9) this diagram contributes

$$\ell_0(z) = \mathrm{gl}(\sigma_2 r(2z)) 2\cos\theta\sin\theta \int_{-\infty<s<t<\infty} \mathrm{d}s\,\mathrm{d}t\,\delta((t+s)\cos\theta)\delta((t-s)\sin\theta) \tag{3.10}$$

for θ the angle of incidence. Here $\mathrm{gl} : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ concatenates the two factors in the tensor product. Letting $u = (t+s)\cos\theta$, $v = (t-s)\sin\theta$ this simplifies to

$$\ell_0(z) = \mathrm{gl}(\sigma_2 r(2z)) \int_{v>0} \mathrm{d}u\,\mathrm{d}v\,\delta(u)\delta(v) = \frac{1}{2}\mathrm{gl}(\sigma_2 r(2z))\,. \tag{3.11}$$

The remaining diagrams describe bulk-boundary interactions, and there as many of these as there are boundary Wilson lines. They are all represented by the diagram below



where z_* is the spectral parameter of a boundary Wilson line. It's easy to see that this contributes

$$\delta\ell_*(z) = 2\pi_{*,2}(r(z - z_*)) \tag{3.12}$$

where denotes π_* projection onto $\mathfrak{p}_*$, the copy of $\mathfrak{p}$ associated to the boundary Wilson line at z_*. To obtain the full solution we should add together the contributions from all of the boundary Wilson lines. The abstract ℓ-matrix is therefore

$$\ell(z) = \ell_0(z) + \sum_{z_*} \delta\ell_*(z) \quad = \frac{1}{2}\mathrm{gl}(\sigma_2 r(2z)) + 2\sum_{z_*} \pi_{*,2}(r(z - z_*))\,, \tag{3.13}$$

and it takes values in $U(\mathfrak{g}) \otimes \bigotimes_{z_*} U(\mathfrak{p}_*)$.

3.2 Trigonometric case

In this section we show how to generate trigonometric K-matrices from CS_4. We also compute the classical ℓ-matrix in a number of cases, matching the results to examples of K-matrices appearing in the literature. We assume familiarity with the boundary conditions for CS_4 used to generate bulk trigonometric R-matrices appearing in [50] and reviewed in section 1.3.5.

Consider CS_4 on the orbifold $\widetilde{\mathcal{V}} = (\Sigma \times C)/\mathbb{Z}_2$ with $(C, \omega) = (\mathbb{C}/2\pi\mathbb{Z}, \mathrm{d}z) \cong (\mathbb{C}^*, \mathrm{d}u/u)$. We use the same boundary conditions on A as in the bulk: A_Σ is required to take values in $\mathfrak{g}_\pm$ as $\mathrm{Re}\, z \to \pm\infty$ for $(\mathfrak{g}, \mathfrak{g}_+ . \mathfrak{g}_-)$ a Manin triple. Since $\mathcal{P}$ exchanges these two limits we must lift it to the adjoint bundle in such a way that it swaps the boundary conditions.

Since the adjoint bundle trivial, i.e., isomorphic to $\mathcal{V} \times \mathfrak{g}$, the lift of $\mathcal{P}$ can be represented by a v-dependent automorphism $\hat{\sigma} : \mathcal{V} \to \mathrm{Aut}\, \mathfrak{g}$. For it to preserve the vacuum $A = 0$, $\hat{\sigma}$ must be constant in Σ and holomorphic in C. For the moment we take it to be independent of z also, so that $\hat{\sigma} = \sigma \in \mathrm{Aut}\, \mathfrak{g}$ obeying $\sigma^2 = \mathrm{id}$..

To ensure compatibility with the boundary conditions we assume that σ exchanges the subalgebras $\mathfrak{g}_\pm$, i.e., that $\sigma(\mathfrak{g}_+) = \mathfrak{g}_-$. To be a symmetry of the action it must also preserve the bilinear tr, which doesn't immediately follow from the fact that it's an automorphism because $\mathfrak{g}$ need not be semisimple. In terms of the basis $\{t_A\}_{A=1}^{\dim \mathfrak{g}_+}$ of $\mathfrak{g}_+$ and the associated dual basis $\{\bar{t}^A\}_{A=1}^{\dim \mathfrak{g}_+}$ of $\mathfrak{g}_-$, the fixed point subalgebra, $\mathfrak{p}$, of σ is spanned by $\{u_A\}_{A=1}^{\dim \mathfrak{g}_+}$ where

$$u_A = \frac{1}{2}(t_A + \sigma(t_A))\,. \tag{3.14}$$

In the trigonometric case $z_* = 0, i\pi$, or equivalently $u_* = \pm 1$. It is therefore natural to view the subscript $*$ as taking values in the set $\{\pm\}$.

It is then straightforward to specialise the ℓ-matrix (3.13) to this case:

$$\begin{aligned}\ell(z) = \frac{1}{2(e^{2z}-1)} c^\sigma_- \otimes 1 \otimes 1 + \frac{e^{2z}}{2(e^{2z}-1)} c^\sigma_+ \otimes 1 \otimes 1 \\ + \frac{2}{e^z - 1} \bar{t}^A \otimes u_A \otimes 1 + \frac{2e^z}{e^z-1} \sigma(\bar{t}^A) \otimes u_A \otimes 1 \\ - \frac{2}{e^z+1} \bar{t}^A \otimes 1 \otimes u_A + \frac{2e^z}{e^z+1} \sigma(\bar{t}^A) \otimes 1 \otimes u_A \,,\end{aligned} \tag{3.15}$$

The three factors in the tensor product correspond to the bulk and two boundary Wilson lines respectively, and

$$\begin{aligned} c^\sigma_- &= \mathrm{gl}(\sigma_1(c_{+,-})) = \sigma(t_A)\bar{t}^A = \bar{t}^A \sigma(t_A)\,, \\ c^\sigma_+ &= \mathrm{gl}(\sigma_1(c_{-,+})) = \sigma(\bar{t}^A) t_A = t_A \sigma(\bar{t}^A)\,. \end{aligned} \tag{3.16}$$

We have used the fact that σ preserves tr in the final equality on both lines.

In section 1.3.5 we saw how to construct Manin triples starting with a reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \widetilde{\mathfrak{h}}$ for $\mathfrak{g}_0$ simple and $\widetilde{\mathfrak{h}}$ a copy of its Cartan. The subalgebras $\mathfrak{g}_\pm$ depended on a choice of Cartan, base and orthogonal matrix $M \in O(\widetilde{\mathfrak{h}})$. (Orthogonal with respect to the restriction of $\mathrm{tr}_{\mathfrak{g}_0}$ to the Cartan.) The classification of involutive automorphisms σ swapping $\mathfrak{g}_\pm$ and preserving the bilinear $\mathrm{tr}_{\mathfrak{g}}$ for a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ of this type follows from standard Lie theory. See appendix A.2 for a full derivation. We find that suitable σ exist only for certain choices of M, and are completely determined by an involutive automorphism of $\mathfrak{g}_0$ which we denote by χ. The data we use in defining χ is an involutive automorphism γ of the Dynkin diagram of $\mathfrak{g}_0$, together with an element $\lambda \in \mathfrak{h}$ which is invariant under the natural action of γ on the Cartan. Explicitly

$$\chi = \exp(\mathrm{ad}_\lambda) \circ \Gamma \circ \omega \tag{3.17}$$

where

$$\omega : (e_\mu, f_\mu, h_\mu) \mapsto (f_\mu, e_\mu, -h_\mu) \tag{3.18}$$

is the Chevalley involution and Γ extends the action of γ to all of $\mathfrak{g}_0$ by

$$\Gamma : (e_\mu, f_\mu, h_\mu) \mapsto (e_{\gamma(\mu)}, f_{\gamma(\mu)}, h_{\gamma(\mu)})\,. \tag{3.19}$$

Here μ is an index for the base Δ. We get an associated solution of the BYBE whenever

$$(M \circ \chi|_{\widetilde{\mathfrak{h}}})^2 = \mathbb{1}_{\widetilde{\mathfrak{h}}}\,. \tag{3.20}$$

where $\chi|_{\widetilde{\mathfrak{h}}}$ is the restriction of χ to the Cartan. The action of σ on $\widetilde{\mathfrak{h}}$ is given by $M \circ \chi|_{\widetilde{\mathfrak{h}}}$. In particular, if γ is the identity then (3.20) implies that M itself is involutive. In this case, since M must not have a $+1$ eigenvalue, it is fixed to be $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$.

3.2.1 Examples of trigonometric K-matrices

We now construct explicit examples of trigonometric ℓ-matrices (without boundary degrees of freedom) for $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. We then match these to the perturbative expansions of K-matrices in the literature which solve the BYBE for the Jimbo trigonometric R-matrix.

Before doing so we must qualify what we mean by 'matching'. Prior to the paper [29] there was no established notion of a quasi-classical K-matrix. As such, all examples of K-matrices in the literature are not formal power series in a parameter $\hbar$ but instead depend on a set of finite parameters. As in subsection 2.2.6 we refer to these as finite K-matrices. Fortunately, it is often possible to identify a sensible classical limit and then introduce $\hbar$ dependence by hand to obtain quasi-classical counterparts. I have included a short summary of this procedure in appendix B.2. Unfortunately there is no guarantee, even starting with a generic finite K-matrix for a given finite R-matrix, that the resulting quasi-classical K-matrix is the most general for the corresponding quasi-classical R-matrix. This prevents us from deducing all orders results as we did for the rational case in subsection 2.2.6. There we bypassed this issue using the uniqueness result for quasi-classical K-matrices proven in subsection 2.3.5. (This is also how the same issue for R-matrices was circumvented in [51].) Unfortunately we have been unable to prove a similar result in the trigonometric case, as we shall discuss in subsection 3.4. This means that we are unable to match the trigonometric K-matrices we compute with those obtained from the literature to all orders in $\hbar$, even up to scale. Instead we show that they are equal at order $\hbar$, and conjecture full agreement. We will see that even this is far from trivial.

Example 3.2.1. We begin by considering $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$ with basis $\{e, f, h, \widetilde{h}\}$. The Dynkin diagram of $\mathfrak{sl}_2(\mathbb{C})$ admits only the trivial diagram automorphism $\gamma = \text{id.}$, so

$$\chi = \exp(\mathrm{ad}_{\lambda h}) \circ \omega \tag{3.21}$$

for some $\lambda \in \mathbb{C}$. It acts by

$$\chi(e) = \Lambda f\,, \qquad \chi(f) = \Lambda^{-1} e\,, \qquad \chi(h) = -h\,, \tag{3.22}$$

where $\Lambda = e^{2\lambda}$. It is inner, with $\chi = \operatorname{conj} \tau$ for

$$\tau = i \begin{pmatrix} 0 & \Lambda^{-1/2} \\ \Lambda^{1/2} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})\,. \tag{3.23}$$

We must have $M = -1$, and so σ acts on $\widetilde{h}$ trivially. The fixed point subalgebra of σ is

$$\mathfrak{p} = \langle \{\Lambda^{-1/2} e + \Lambda^{1/2} f, \widetilde{h}\} \rangle_{\mathbb{C}} \cong \mathbb{C}^2\,. \tag{3.24}$$

Boundary Wilson lines at $u = e^z = \pm 1$ are labelled by representations of this algebra.

We need to make appropriate choices of representations $V, W_\pm$ for the bulk and boundary Wilson lines. Let V be the tensor product of the fundamental of $\mathfrak{sl}_2(\mathbb{C})$ with the 1 dimensional representation $\widetilde{h} \mapsto s \in \mathbb{C}$ of $\widetilde{\mathfrak{h}}$. In order to obtain K-matrices without boundary degrees of freedom the boundary Wilson lines at $u = \pm 1$ are taken to be in the 1 dimensional representations

$$\Lambda^{-1/2} e + \Lambda^{1/2} f \mapsto q_\pm \in \mathbb{C}\,, \qquad \widetilde{h} \mapsto r_\pm \in \mathbb{C} \tag{3.25}$$

respectively.

Then the classical ℓ-matrix (3.15) evaluates to

$$\begin{aligned} \ell_{V\otimes W_+ \otimes W_-}(z) = \Bigg(& -\frac{1}{8}\frac{e^{2z}+1}{e^{2z}-1}(1-s^2) + \frac{1}{2}\frac{e^z+1}{e^z-1} r_+ s + \frac{1}{2}\frac{e^z-1}{e^z+1} r_- s \Bigg) \mathbb{1}_V \\ & + \Lambda^{-1/2} \left(q_+ \frac{e^z}{e^z-1} + q_- \frac{e^z}{e^z+1} \right) E \\ & \quad + \Lambda^{1/2} \left(q_+ \frac{1}{e^z-1} - q_- \frac{1}{e^z+1} \right) F \\ & \qquad - \frac{i}{4}(2r_+ + 2r_- + s) H\,. \end{aligned} \tag{3.26}$$

Here E, F, H are the images of e, f, h in the fundamental representation of $\mathfrak{sl}_2(\mathbb{C})$. To first order in $\hbar$ this is consistent with

$$K(z) \sim \begin{pmatrix} \hbar \hat{q} e^{z/2} \sinh(\hat{\xi}/2 + z/2) & \hat{\Lambda}^{-1/2} \sinh z \\ \hat{\Lambda}^{1/2} \sinh z & \hbar \hat{q} e^{-z/2} \sinh(\hat{\xi}/2 - z/2) \end{pmatrix}\,, \tag{3.27}$$

for $(\hat{q}, \hat{\xi}, \hat{\Lambda}) \in \mathbb{C}[[\hbar]]$ given by

$$\hat{q} = q(1+\mathcal{O}(\hbar)), \qquad \hat{\xi} = \xi(1+\mathcal{O}(\hbar)), \qquad \hat{\Lambda} = \Lambda(1+i\hbar(r_+ + r_- + s/2) + \mathcal{O}(\hbar^2)) \quad (3.28)$$

and q, ξ defined by $q_+ = q\sinh(\xi/2)$, $q_- = q\cosh(\xi/2)$.

For ease of comparison with the literature it's convenient to first symmetrize the R-matrix, conjugating it by

$$C(z) = i\begin{pmatrix} 0 & e^{z/4} \\ e^{-z/4} & 0 \end{pmatrix} \quad (3.29)$$

as discussed in [50]. The corresponding transformation of the K-matrix is

$$K(z) \mapsto C(z)K(z)C(-z)^{-1}, \quad (3.30)$$

which gives

$$K(z) \sim \begin{pmatrix} \hbar\hat{q}\sinh(\hat{\xi}/2 - z/2) & \hat{\Lambda}^{1/2}\sinh z \\ \hat{\Lambda}^{-1/2}\sinh z & \hbar\hat{q}\sinh(\hat{\xi}/2 + z/2) \end{pmatrix}. \quad (3.31)$$

On the right hand side is the most general finite K-matrix for the corresponding R-matrix appropriately interpreted as a quasi-classical K-matrix with classical limit τ_V [82, 57].

Example 3.2.2. As a second example, we consider $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}) \oplus \mathbb{C}^2$. We restrict our attention to involutive automorphisms for which χ is inner. Since the Chevalley involution of $\mathfrak{sl}_3(\mathbb{C})$ is outer, we choose γ to be the non-trivial automorphism of the Dynkin diagram. This swaps the two simple roots, i.e., $\gamma = (1\,2)$. So

$$\chi = \mathrm{ad}_{\lambda(h_1+h_2)} \circ \Gamma \circ \omega \quad (3.32)$$

for some $\lambda \in \mathbb{C}$. Acting on the Chevalley generators $\{e_i, f_i, h_i\}_{i\in\{1,2\}}$ of $\mathfrak{sl}_3(\mathbb{C})$, χ is given by

$$h_1 \mapsto -h_2, \qquad e_1 \mapsto \Lambda^{-1} f_2, \qquad e_2 \mapsto \Lambda^{-1} f_1, \quad (3.33)$$

where $\Lambda = e^{\lambda}$. The action of χ on the remaining generators is determined by the fact it is involutive. It can be realised as conjugation by the matrix

$$\tau = -\begin{pmatrix} 0 & 0 & \Lambda \\ 0 & 1 & 0 \\ \Lambda^{-1} & 0 & 0 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}). \quad (3.34)$$

Since γ is no longer the identity there exist non-trivial $M \in O(\widetilde{\mathfrak{h}})$ for which

$$(M \circ \chi|_{\widetilde{\mathfrak{h}}})^2 = \mathbb{1}_{\widetilde{\mathfrak{h}}} \,. \tag{3.35}$$

In this example, the above constraint fixes M to be a complex rotation. This includes the possibility that $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$ and, for the sake of simplicity, we concentrate on this case leading to the standard trigonometric R-matrix. Under this assumption the action of σ on $\widetilde{\mathfrak{h}}$ is

$$\sigma(\widetilde{h}_1) = \widetilde{h}_2 \,. \tag{3.36}$$

The $+1$ eigenspace of σ is therefore

$$\begin{aligned} \mathfrak{p} &= \Big\langle \Big\{ \Lambda^{1/2} e_1 + \Lambda^{-1/2} f_2 \,, \Lambda^{1/2} e_2 + \Lambda^{-1/2} f_1 \,, \Lambda e_\gamma + \Lambda^{-1} f_\gamma \,, h_2 - h_1, \widetilde{h}_1 + \widetilde{h}_2 \Big\} \Big\rangle_{\mathbb{C}} \\ &\cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2 \,. \end{aligned} \tag{3.37}$$

Here $\gamma = \alpha_1 + \alpha_2$ for α_1 and α_2 the simple roots of $\mathfrak{sl}_3(\mathbb{C})$, so $e_\gamma = [e_1, e_2]$ and $f_\gamma = [f_2, f_1]$. The two abelian summands in $\mathfrak{p}$ are generated by

$$h_2 - h_1 + 3\Lambda e_\gamma + 3\Lambda^{-1} f_\gamma \,, \qquad \widetilde{h}_1 + \widetilde{h}_2 \,. \tag{3.38}$$

Now we make choices for the representations $V, W_\pm$ for the bulk and boundary Wilson lines. Let V be the fundamental representation of $\mathfrak{sl}_3(\mathbb{C})$ with no charge under the $\widetilde{\mathfrak{h}}$ factor. We take $W_\pm$ to be 1 dimensional representations of $\mathfrak{p}$ labelled by the charges

$$h_2 - h_1 + 3\Lambda e_\gamma + 3\Lambda^{-1} f_\gamma \mapsto q_\pm \in \mathbb{C} \,, \qquad \widetilde{h}_1 + \widetilde{h}_2 \mapsto r_\pm \in \mathbb{C} \,. \tag{3.39}$$

We can then evaluate the image of the trigonometric ℓ-matrix (3.15) in the representation $V \otimes W_+ \otimes W_-$. This gives

$$\ell_{V\otimes W_+ \otimes W_-}(z) = \begin{pmatrix} -\mathcal{H}(z) + r & 0 & e^z \mathcal{E}(z) \\ 0 & 2\mathcal{H}(z) & 0 \\ e^{-z}\mathcal{F}(z) & 0 & -\mathcal{H}(z) - r \end{pmatrix} , \tag{3.40}$$

where $2ir = r_+ + r_-$ and

$$\begin{aligned}\mathcal{E}(z) &= \frac{\Lambda(1 + q_+ e^{-z/2}\cosh(z/2) + q_- e^{-z/2}\sinh(z/2))}{4\sinh z}\,,\\ \mathcal{F}(z) &= \frac{\Lambda^{-1}(1 + q_+ e^{z/2}\cosh(z/2) - q_- e^{z/2}\sinh(z/2))}{4\sinh z}\,,\\ \mathcal{H}(z) &= \frac{\cosh z + q_+\cosh^2(z/2) + q_-\sinh^2(z/2)}{12\sinh z}\,.\end{aligned} \tag{3.41}$$

Introducing parameters q_1 and q_2 defined by $q_+ = 4q_1 + 4q_2 - 1$ and $q_- = 4q_1 - 4q_2 - 1$ we can simplify these functions to

$$\mathcal{E}(z) = \frac{\Lambda(q_1 + q_2 e^{-z})}{\sinh z}\,, \qquad \mathcal{F}(z) = \frac{\Lambda^{-1}(q_1 + q_2 e^{z})}{\sinh z}\,, \qquad \mathcal{H}(z) = \frac{q_1\cosh z + q_2}{3\sinh z}\,. \tag{3.42}$$

To first order in $\hbar$ this is consistent with

$$K(z) \sim \begin{pmatrix} \hat{\alpha}e^z + \hat{\beta} & 0 & \hat{\delta}\sinh z \\ 0 & \hat{\gamma}\sinh z + \hat{\alpha}\cosh z + \hat{\beta} & 0 \\ \hat{\epsilon}\sinh z & 0 & \hat{\alpha}e^{-z} + \hat{\beta} \end{pmatrix}, \tag{3.43}$$

for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\epsilon} \in \mathbb{C}[[\hbar]]$ given by

$$\begin{gathered}\hat{\alpha} = \hbar q_1 + \mathcal{O}(\hbar^2)\,, \quad \hat{\beta} = \hbar q_2 + \mathcal{O}(\hbar^2)\,, \quad \hat{\gamma} = 1 + \mathcal{O}(\hbar^2)\,,\\ \hat{\delta} = \Lambda(1 + \hbar r + \mathcal{O}(\hbar^2))\,, \quad \hat{\epsilon} = \Lambda^{-1}(1 - \hbar r + \mathcal{O}(\hbar^2))\,.\end{gathered} \tag{3.44}$$

These parameters obey the constraint

$$\hat{\delta}\hat{\epsilon} = \hat{\gamma}^2 - \hat{\beta}^2 \tag{3.45}$$

up to order $\hbar$.

On the right hand side of (3.43), for parameters obeying (3.45), is the most general finite K-matrix for the corresponding R-matrix appropriately interpreted as a quasi-classical K-matrix with classical limit τ_V. This can be seen by analysing the trigonometric K-matrices for $\mathfrak{sl}_3(\mathbb{C})$ in [107]. It's associated to one of three families, but conspicuously we miss the remaining two. Studying these solutions carefully one finds that the analogue of the classical limit for these K-matrices is not z-independent. This motivates us to consider a generalisation of the construction presented thus far.

Remark. The classical ℓ-matrices derived in examples 3.2.1 & 3.2.2 are independent of $\hbar$ in the sense introduced in subsection 2.2.6. We commented there that this is not a generic feature of K-matrices, but it's nonetheless curious to observe this behaviour in the trigonometric case. It seems that all K-matrices solving the BYBE for R-matrices associated to the fundamental of $\mathfrak{sl}_n(\mathbb{C})$ have this property.

3.2.2 Generalisation to z-dependent automorphisms

To accommodate K-matrices with z-dependent classical limits, we propose a considerable generalisation of the construction outlined in chapter 2.

We choose to lift the action of $\mathcal{P} : (x, y, z, \bar{z}) \mapsto (-x, y, -z, -\bar{z})$ to the adjoint bundle of the gauge theory not by simultaneously acting on its fibres with a constant involutive automorphism, but instead by using a v-dependent map $\hat{\sigma} : \mathcal{V} \to \operatorname{Aut}\mathfrak{g}$. To preserve the vacuum $\mathrm{d}' = \mathrm{d}_\Sigma + \bar{\partial}_C$ it must be constant in Σ and holomorphic in C. It is therefore determined by a holomorphic map $C \to \operatorname{Aut}\mathfrak{g}$ which we also denote by $\hat{\sigma}$. For the lift to be involutive we must have $\hat{\sigma}(z)\hat{\sigma}(-z) = \mathbb{1}_{\mathfrak{g}}$. We also require that $\hat{\sigma}(z)$ preserve the bilinear tr on the fibres.

In the trigonometric case we still require that $\hat{\sigma}$ exchanges the boundary conditions as $\operatorname{Re} z \to \pm\infty$. It must also not introduce poles when acting on non-singular field configurations. In appendix A.3 we discuss how suitable $\hat{\sigma}$ can be constructed from automorphisms of the loop algebra $L\mathfrak{g}$ of the Manin triple $(\mathfrak{g} = \mathfrak{g}_0 + \widetilde{\mathfrak{h}}, \mathfrak{g}_+, \mathfrak{g}_-)$.

There are elements of this proposal that we do not fully understand. In particular, the interplay between the framing anomaly for curved Wilson lines and the action of the z-dependent automorphism $\hat{\sigma}$ seems to be rather subtle. Despite this, there is strong evidence that this approach is correct.

One important check on its validity is whether the R-matrix obeys

$$R_{12}(z) = \hat{\sigma}_1(z)\hat{\sigma}_2(0)R_{21}(z)\,, \tag{3.46}$$

which should be a consequence of the classical symmetry $A \mapsto \hat{\sigma}\mathcal{P}^*A$ of the bulk theory. A failure would signal that the symmetry is anomalous, which would certainly prohibit us from orbifolding by it. Direct calculation shows that it does indeed hold for the trigonometric R-matrix appearing in example 1.3.4 and the choices of $\hat{\sigma}$ used in 3.2.3.

Further evidence is provided in the next subsection, in which we demonstrate the families of trigonometric K-matrices for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ which were missing above can be recovered using this prescription.

To compute the next-to-leading order contribution to the K-matrix for a z-dependent automorphism $\sigma(z)$ we proceed exactly as before. In holomorphic gauge, the non-vanishing component of the propagator is

$$P_{xy}(v_1,v_2)=r(z_1-z_2)\delta(x_1-x_2)\delta(y_1-y_2)+\sigma_2(z_2)(r(z_1+z_2))\delta(x_1+x_2)\delta(y_1-y_2)\,, \tag{3.47}$$

the only difference being that σ is now z-dependent. We find that the classical ℓ-matrix is

$$\ell(z)=\frac{1}{2}\mathrm{gl}(\sigma_2(z)(r(2z))+2\sum_{z_*}\pi_{*,2}(r(z-z_*))\,, \tag{3.48}$$

with the two terms arising from the self-interaction of the bulk Wilson line and interactions between the bulk and boundary Wilson lines respectively. π_* is the projection onto $\mathfrak{p}_*$, the fixed point subalgebra of $\hat{\sigma}(z_*)$. Note that $\mathfrak{p}_*$ need not be isomorphic for different values of z_*.

3.2.3 Examples of trigonometric K-matrices associated to z-dependent automorphisms

Here we construct explicit examples of trigonometric ℓ-matrices associated to z-dependent automorphisms $\hat{\sigma}$ for $\mathfrak{g}_0=\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. We then match these to the perturbative expansions of K-matrices appearing in the literature with the same caveats as in subsection 3.2.1. The z-dependent automorphisms $\hat{\sigma}$ we use here are obtained in appendix A.3 by considering the Loop algebra $L\mathfrak{g}$.

Example 3.2.3. We revisit the case $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}$, which admits a z-dependent automorphism acting on the $\mathfrak{sl}_2(\mathbb{C})$ summand by

$$\hat{\sigma}(z)(e)=e^{z}e\,,\qquad \hat{\sigma}(z)(f)=e^{-z}f\,,\qquad \hat{\sigma}(z)(h)=h\,. \tag{3.49}$$

This is inner in the sense it can be realised as conjugation by

$$\tau(z)=\begin{pmatrix}e^{z/2}&0\\0&e^{-z/2}\end{pmatrix}\in\mathrm{SL}_2(\mathbb{C})\,. \tag{3.50}$$

This automorphism corresponds to the non-trivial transposition of the Dynkin diagram of the affine algebra $\widehat{\mathfrak{sl}_2(\mathbb{C})}$ (see appendix A.3). It extends to act on $\widetilde{\mathfrak{h}}\cong\mathbb{C}$ as $\widetilde{h}\mapsto-\widetilde{h}$.

The subalgebras $\mathfrak{p}_\pm$ of $\mathfrak{g}$ that survive at the fixed points $u=\pm1$ are isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and $\mathbb{C}=\langle h\rangle_{\mathbb{C}}$ respectively. (By replacing z by $z+i\pi$ in $\tau(z)$ we can swap the subalgebras at $u=\pm1$, which will generate a K-matrix essentially equivalent to the one constructed here.)

We take the representation V of the reflecting bulk Wilson line to be the tensor product of the fundamental of $\mathfrak{sl}_2(\mathbb{C})$ with the trivial representation of the abelian summand. We cannot insert a boundary Wilson line at $u = +1$ since there are no 1 dimensional representations of $\mathfrak{p}_+ \cong \mathfrak{sl}_2(\mathbb{C})$, however at $u = -1$ we choose W_- to be the 1 dimensional representation with charge $h \mapsto q \in \mathbb{C}$.

The corresponding classical ℓ-matrix is then

$$\ell_{V\otimes 1\otimes W_-}(z) = \frac{\coth(z/2)}{8}\mathbb{1}_V + \frac{q\tanh(z/2)}{2}H\,, \tag{3.51}$$

which to first order in $\hbar$ is consistent with

$$K(z) \sim \begin{pmatrix} e^{z/2}\cosh((\hbar\hat{q}+z)/2) & 0 \\ 0 & e^{-z/2}\cosh((\hbar\hat{q}-z)/2) \end{pmatrix} \tag{3.52}$$

for $\hat{q} = q(1+\mathcal{O}(\hbar)) \in \mathbb{C}[[\hbar]]$. To symmetrise the R-matrix we conjugate it by

$$C(z) = i\begin{pmatrix} 0 & e^{z/4} \\ e^{-z/4} & 0 \end{pmatrix}\,, \tag{3.53}$$

under which $K(z)$ transforms by $K(z) \mapsto C(z)K(z)C^{-1}(-z)$ giving

$$K(z) \sim \begin{pmatrix} \sinh((i\pi+\hbar\hat{q}-z)/2) & 0 \\ 0 & \sinh(i\pi+\hbar\hat{q}+z)/2) \end{pmatrix}\,. \tag{3.54}$$

On the right hand side is the most general finite K-matrix for the corresponding R-matrix appropriately interpreted as a quasi-classical K-matrix with classical limit $\tau_V(z)$ [56]. We believe it corresponds to the same family of finite K-matrices as those generated in example 3.2.1, however, this solution is expanded around a different classical limit.

Example 3.2.4. Next we take $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}) \oplus \mathbb{C}^2$. Consider the z-dependent automorphism acting on the $\mathfrak{sl}_3(\mathbb{C})$ summand by

$$\begin{gathered} \hat{\sigma}(z)(e_1) = \Lambda e^z e_\gamma\,, \qquad \hat{\sigma}(z)(e_2) = \Lambda^{-2} f_2\,, \qquad \hat{\sigma}(z)(f_1) = \Lambda^{-1}e^{-z}f_\gamma\,, \\ \hat{\sigma}(z)(h_1) = h_1 + h_2\,, \qquad \hat{\sigma}(z)(h_2) = -h_2\,. \end{gathered} \tag{3.55}$$

The action of $\hat{\sigma}$ on the remaining generators is determined by $\hat{\sigma}(z)\hat{\sigma}(-z) = \mathbb{1}_{\mathfrak{g}}$. This is inner in the sense that it can be realised as conjugation by

$$\tau(z) = e^{i\pi/3} \begin{pmatrix} e^{2z/3} & 0 & 0 \\ 0 & 0 & \Lambda e^{-z/3} \\ 0 & \Lambda^{-1} e^{-z/3} & 0 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C})\,. \tag{3.56}$$

It corresponds to the non-trivial transposition of the Dynkin diagram of the affine algebra $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ that swaps the 0^{th} and 1^{st} simple roots. A similar automorphism exists for the transposition swapping the 0^{th} and 2^{nd} roots. Under the simplifying assumption that $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$, $\hat{\sigma}$ extends to the whole of $\mathfrak{g}$ by acting on $\widetilde{\mathfrak{h}}$ as $\widetilde{h}_1 \mapsto -\widetilde{h}_1 - \widetilde{h}_2$ and $\widetilde{h}_2 \mapsto \widetilde{h}_2$. The subalgebras $\mathfrak{p}_\pm$ are both isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2$ with one abelian summand generated by

$$2h_1 + h_2 \pm (3\Lambda e_2 + 3\Lambda^{-1} f_2)\,. \tag{3.57}$$

and the other by $\widetilde{h}_2$. We take the representation V of the reflecting bulk Wilson line to be the fundamental representation of $\mathfrak{sl}_3(\mathbb{C})$ tensored with the trivial representation of $\widetilde{\mathfrak{h}}$. The 1 dimensional representations $W_\pm$ of the boundary Wilson lines at $u = \pm 1$ are labelled by charges

$$2h_1 + h_2 \pm (3\Lambda e_2 + 3\Lambda^{-1} f_2) \mapsto q_\pm\,, \qquad \widetilde{h} \mapsto r_\pm\,. \tag{3.58}$$

The classical ℓ-matrix is then

$$\ell_{V\otimes W_+ \otimes W_-}(z) = \begin{pmatrix} 2\mathcal{H}(z) & 0 & 0 \\ 0 & -\mathcal{H}(z) + r & \mathcal{F}(z) \\ 0 & \mathcal{E}(z) & -\mathcal{H}(z) - r \end{pmatrix}, \tag{3.59}$$

for $2ir = r_+ + r_-$ and $\mathcal{E}(z), \mathcal{F}(z)$ and $\mathcal{H}(z)$ given in equation (3.41). As discussed there, these functions can be simplified by introducing the parameters q_1 and q_2 defined by $q_+ = q_1 + q_2 - 1$ and $q_- = q_1 - q_2 - 1$. The resulting expressions are given in equation (3.42).

To first order in $\hbar$ this is consistent with

$$K(z) \sim \begin{pmatrix} e^z(\hat{\gamma}\sinh z + \hat{\alpha}\cosh z + \hat{\beta}) & 0 & 0 \\ 0 & \hat{\alpha} + \hat{\beta} e^z & \hat{\delta}\sinh z \\ 0 & \hat{\epsilon}\sinh z & \hat{\alpha} + \hat{\beta} e^{-z} \end{pmatrix}. \tag{3.60}$$

for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\epsilon} \in \mathbb{C}[[\hbar]]$ given by

$$\begin{aligned} \hat{\alpha} = \hbar q_1 + \mathcal{O}(\hbar^2)\,, \quad \hat{\beta} = \hbar q_2 + \mathcal{O}(\hbar^2)\,, \quad \hat{\gamma} = 1 + \mathcal{O}(\hbar^2)\,, \\ \hat{\delta} = \Lambda(1 + \hbar r + \mathcal{O}(\hbar^2))\,, \quad \hat{\epsilon} = \Lambda^{-1}(1 - \hbar r + \mathcal{O}(\hbar^2))\,. \end{aligned} \tag{3.61}$$

These parameters obey the constraint

$$\hat{\delta}\hat{\epsilon} = \hat{\gamma}^2 - \hat{\alpha}^2 \tag{3.62}$$

up to first order in $\hbar$. On the right hand side of (3.60), for parameters obeying (3.62), is the most general finite K-matrix for the corresponding R-matrix appropriately interpreted as a quasi-classical K-matrix with classical limit $\tau_V(z)$. This can be seen by analysing the trigonometric K-matrices for $\mathfrak{sl}_3(\mathbb{C})$ in [107]. It belongs to one of the two families of K-matrices appearing therein which were missing from our analysis in example 3.2.2.

Choosing

$$\tau(z) = e^{i\pi/3} \begin{pmatrix} 0 & \Lambda e^{z/3} & 0 \\ \Lambda^{-1} e^{z/3} & 0 & 0 \\ 0 & 0 & e^{-2z/3} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}) \tag{3.63}$$

allows us to generate the final family of solutions.

Remark. By introducing non-trivial bundle morphisms $\hat{\sigma}$ we have conjecturally obtained all quasi-classical trigonometric K-matrices for $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. (Here we mean solutions to the BYBE for the quasi-classical analogue of Jimbo's trigonometric R-matrix [93].) Certainly we have found counterparts of all finite K-matrices of this type [56, 107]. It is natural to ask if we can hope to obtain all quasi-classical trigonometric K-matrices in this way. Unfortunately this does not seem to be the case. Recent progress in the study of quantum symmetric pairs would seem to suggest that trigonometric K-matrices are associated to automorphisms of quantum affine algebras of the 'second kind' [97, 99, 148]. The z-dependent automorphisms constructed in appendix A.3 are associated to a strict subset of such automorphisms. Those that are missing do not seem to completely exchange the boundary conditions as $\mathrm{Re}\, z \to \pm\infty$. It would be interesting to incorporate these into CS_4.

3.3 Elliptic case

In this section we show how to generate elliptic K-matrices using CS_4. We also compute the elliptic ℓ-matrix in the fundamental representation, and match it to K-matrices from the literature. We assume familiarity with the choice of G-bundle and vacuum used to generate elliptic R-matrices using CS_4. These were introduced in [50] and are reviewed in section 1.3.5.

Let $\widetilde{\mathcal{V}} = (\Sigma \times C)/\mathbb{Z}_2$ for $(C, \omega) = (\mathrm{T}^2_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \mathrm{d}z)$. To define CS_4 on the orbifold we must lift the action of $\mathcal{P}$ to the adjoint bundle of the gauge theory in such a way that it preserves its topology and the vacuum. We write $\hat{\sigma}$ for this lift.

Recall that in the elliptic case we are obliged to take $G = \mathrm{PSL}_n(\mathbb{C})$. The adjoint bundle of the theory is the pullback of a $\mathfrak{g}$-bundle $\mathrm{Ad}\, P$ over T^2_τ, and the vacuum partially holomorphic structure is the pullback of a holomorphic structure on $\mathrm{Ad}\, P$. In order for $\hat{\sigma}$ to preserve the vacuum it must be constant in Σ and so is determined by a lift of the map $z \mapsto -z$ on T^2_τ to $\mathrm{Ad}\, P$. We abuse notation by also writing $\hat{\sigma}$ for this lift. We therefore seek a bundle map $\hat{\sigma} : \mathrm{Ad}\, P \to \mathrm{Ad}\, P$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Ad}\, P & \xrightarrow{\hat{\sigma}} & \mathrm{Ad}\, P \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{T}^2_\tau & \xrightarrow{z \mapsto -z} & \mathrm{T}^2_\tau \end{array} \tag{3.64}$$

For this lift to preserve the vacuum, $\hat{\sigma}$ must be holomorphic. Since $z \mapsto -z$ reverses the directions of the cycles on T^2_τ, it must also invert the corresponding holonomies.

In appendix A.4 we classify the allowed bundle morphisms and show that they depend on two discrete parameters $\xi, \eta \in \mathbb{Z}_n$. Viewing $\mathrm{Ad}\, P$ as the quotient of $\mathbb{C} \times \mathfrak{sl}_n(\mathbb{C})$ by the equivalence relation $(z, X) \sim (z + a + b\tau, \mathrm{conj}(A^{-\zeta a} B^{-b})(X))$, they are given by

$$\hat{\sigma} : \mathbb{C} \times \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C} \times \mathfrak{sl}_n(\mathbb{C}) , \qquad (z, X) \mapsto (-z, \sigma(X)) \tag{3.65}$$

for

$$\sigma(t_{i,j}) = \epsilon^{i\xi + j\eta} t_{-i,-j} \, . \tag{3.66}$$

Note that $\sigma = \mathrm{conj}\, \tau$ for

$$\tau^\alpha{}_\beta = \epsilon^{-\zeta\xi\alpha} \delta_{\alpha+\beta+\eta \equiv 0\, (n)} \, , \tag{3.67}$$

and is therefore inner.

The quotient $\mathrm{T}^2_\tau/\{z \sim -z\}$ has four conical singularities corresponding to the four fixed points, and embeds in 3 dimensional space as a ‘pillowcase’, illustrated in figure

3.1. We have included the image of a B cycle on the pillowcase. Note that it intersects the image of an A cycle twice on account of the orbifolding. For each of the four corners of the pillowcase there is a corresponding line of orbifold singularities in $\widetilde{\mathcal{V}}$: $L_{z_*} = \{x = 0, z = z_*\}$ for $z_* = (a_* + b_*\tau)/2$ with $a_*, b_* \in \{0, 1\}$. Let us consider the behaviour of the partial connection along these lines.

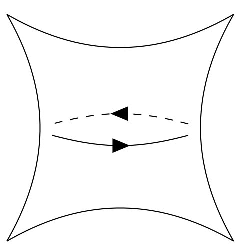


Fig. 3.1 Pillowcase orbifold

We can view the partial connection as a section of $\Sigma \times (\mathbb{C} \times \mathfrak{sl}_n(\mathbb{C}))$ with quasi-periodicities

$$A(x, y, z + a + b\tau, \bar{z} + a + b\bar{\tau}) = \mathrm{conj}(A^{a\zeta})\mathrm{conj}(B^b)A(x, y, z, \bar{z}) \tag{3.68}$$

for $a, b \in \mathbb{Z}$, and obeying the orbifold condition $A = \sigma\mathcal{P}^*A$. Pulling back to the representative of z_* in the fundamental domain of $\mathbb{C}$ we have

$$\iota^*_{L_{z_*}}A = \sigma\Big(\iota^*_{L_{-z_*}}A\Big) = \sigma_*\Big(\iota^*_{L_{z_*}}A\Big)\,, \tag{3.69}$$

where we have defined $\sigma_* = \sigma \circ \mathrm{conj}(A^{-\zeta a_*}B^{-b_*})$.[2] This shows that the subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ in which $\iota_{L_{z_*}}A$ takes values depends on which of the four singular lines we pull back to. Since

$$\sigma_*(t_{i,j}) = \epsilon^{i(\xi - a_*) + j(\eta - b_*)}t_{-i,-j}\,, \tag{3.70}$$

the effect of the correction $\sigma \mapsto \sigma_*$ is to shift

$$\xi \mapsto \xi - a_* = \xi_*\,, \qquad \eta \mapsto \eta - b_* = \eta_*\,. \tag{3.71}$$

The positive eigenspace of σ_*, which we denote by $\mathfrak{p}_*$, depends on n, ξ_*, and η_*. We have the following possibilities:

- If n is odd then $\dim \mathfrak{p}_* = (n^2 - 1)/2$ and $\mathfrak{p}_* \cong \mathfrak{sl}_{(n+1)/2}(\mathbb{C}) \oplus \mathfrak{sl}_{(n-1)/2}(\mathbb{C}) \oplus \mathbb{C}$.

[2] The ambiguity in the lift of $z \mapsto -z$ to $\mathbb{C}$ and the choice of representative for z_* can be absorbed into a different choice of σ.

- If n, ξ_*, and η_* are all even then $\dim \mathfrak{p}_* = n^2/2 + 1$ and $\mathfrak{h}_* \cong \mathfrak{sl}_{n/2+1}(\mathbb{C}) \oplus \mathfrak{sl}_{n/2-1}(\mathbb{C}) \oplus \mathbb{C}$.
- If n is even and either ξ_* or η_* is odd then $\dim \mathfrak{p}_* = n^2/2 - 1$ and $\mathfrak{p}_* \cong \mathfrak{sl}_{n/2}(\mathbb{C}) \oplus \mathfrak{sl}_{n/2}(\mathbb{C}) \oplus \mathbb{C}$.

We can see that for n odd the subalgebras on the singular lines are all isomorphic, but for n even the subalgebra on one of the four lines is not isomorphic to the other three. In all cases $\mathfrak{p}_*$ contains an abelian summand. By inserting boundary Wilson lines in 1 dimensional representations of these abelian summands along the corresponding singular lines we will be able to generate continuous families of K-matrices without boundary degrees of freedom.

3.3.1 Examples of elliptic K-matrices

We now construct explicit examples of elliptic ℓ-matrices without boundary degrees of freedom. We then match these to the perturbative expansions of K-matrices appearing in the literature. This matching is subject to the same caveats as in the trigonometric case, see the discussion in subsection 3.2.1 for details. We explain where the uniqueness proof for rational K-matrices fails in the elliptic case in section 3.4.

We are obliged to take $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\hat{\sigma}$ to be the bundle map defined in equation (3.66). Note that although the lift of $\mathcal{P}$ to the adjoint bundle is non-trivial, we can still apply the formula (3.13) for the ℓ-matrix since in the trivialisation we use $\hat{\sigma}$ is represented by a constant involutive automorphism σ of $\mathfrak{sl}_n(\mathbb{C})$.

We treat n even and odd separately since they are qualitatively rather different.

Example 3.3.1. In this first example we take n odd. The fixed point subalgebras of σ_* are all isomorphic: $\mathfrak{p}_* \cong \mathfrak{sl}_{(n+1)/2} \oplus \mathfrak{sl}_{(n-1)/2} \oplus \mathbb{C}$. In terms of n, ξ_*, and η_* the generators of the abelian summands are

$$Q_* = \frac{1}{n} \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-2^{-1}(\zeta^{-1}ij+\xi_* i+\eta_* j)} t_{i,j}\,. \tag{3.72}$$

Here 2^{-1} is the inverse of 2 modulo n which exists since n is odd.

We now fix representations V, W_* for the bulk reflecting Wilson line and four boundary Wilson lines respectively. To get solutions of the BYBE for Belavin's symmetric R-matrix [20] we take $V = U$ to be the fundamental of $\mathfrak{sl}_n(\mathbb{C})$, and the 1 dimensional representations W_* to be labelled by charges $Q_* \mapsto \tilde{q}_* \in \mathbb{C}$.

From (3.11), the self-interactions of the reflecting bulk Wilson line contribute

$$\begin{aligned}\ell_0(z) &= \frac{1}{2n} \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-\zeta^{-1}ij} w_{i,j}(2z) t_{i,j} \sigma_*(t_{-i,-j}) \,, \\ &= \frac{1}{2n} \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-(\zeta^{-1}ij+\xi i+\eta j)} w_{i,j}(2z) t_{i,j}^2 \,.\end{aligned} \tag{3.73}$$

to the classical ℓ-matrix. Recall that we write $T_{i,j}$ for the images of the basis elements $t_{i,j}$ in the representation U, and extend them to a basis of $\operatorname{End} U$ by adjoining $T_{0,0} = \mathbb{1}_U$. Then

$$\ell_{0,U}(z) = \frac{1}{2n} \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-(2\zeta^{-1}ij+\xi i+\eta j)} w_{i,j}(2z) T_{2i,2j} \,, \tag{3.74}$$

where we've used the fact that $T_{i,j}^2 = \epsilon^{-\zeta^{-1}ij} T_{2i,2j}$.

Similarly, from (3.12), bulk-boundary interactions contribute

$$\sum_{z_*} \delta\ell_{*,U\otimes W_*}(z) = \frac{2}{n^2-1} \sum_{z^*} \tilde{q}_* \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-2^{-1}(\zeta^{-1}ij+\xi_* i+\eta_* j)} w_{i,j}(z-z^*) T_{i,j} \,. \tag{3.75}$$

By making the change of variables

$$i \mapsto 2^{-1} i \,, \qquad j \mapsto 2^{-1} j \tag{3.76}$$

in the sum in equation (3.74) and using the identity

$$w_{2^{-1}i,2^{-1}j}(2z) = \frac{1}{2} \sum_{a_*,b_*\in\{0,1\}} \epsilon^{ia_*+jb_*} w_{i,j}(z-z_*) \tag{3.77}$$

we can rewrite it as

$$\ell_{0,U}(z) = \frac{1}{4n} \sum_{a_*,b_*\in\{0,1\}} \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-2^{-1}(\zeta^{-1}ij+\xi_* i+\eta_* j)} w_{i,j}(z-z_*) T_{i,j} \,. \tag{3.78}$$

This contribution can be absorbed into (3.75) by shifting the parameters

$$q_* = \left(\frac{1}{4} + \frac{2n}{n^2-1}\tilde{q}_*\right) \,. \tag{3.79}$$

Therefore the full classical ℓ-matrix is

$$\ell_{U\otimes W}(z) = \frac{1}{n} \sum_{z^*} q_* \sum_{(i,j)\in\mathcal{I}_n} \epsilon^{-2^{-1}(\zeta^{-1}ij+\xi_* i+\eta_* j)} w_{i,j}(z-z_*) T_{i,j} \,, \tag{3.80}$$

where we have abbreviated $W = \bigotimes_* W_*$. In appendix A.5 we show that, for $\zeta = 1$ and $\xi = \eta = 0$, to first order in $\hbar$ this is consistent with

$$K(z) \sim \left(\sum_{z_*} \hat{\mu}_* \sum_{i,j \in \mathbb{Z}_n^2} L_*^{i,j}(z, \hat{\Lambda}) T_{i,j} \right) \tau_U \tag{3.81}$$

for

$$\begin{aligned} &L_*^{i,j}(z; \hat{\Lambda}) \\ &\quad = \epsilon^{-2^{-1}(ij - a_* i - b_* j)} e^{2\pi i b_* \hat{\Lambda}/n} \frac{\theta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](-2\hat{\Lambda}/n|\tau)\theta\left[\begin{smallmatrix} 1/2+i/n \\ 1/2-j/n \end{smallmatrix}\right](z - z_* - 2\hat{\Lambda}/n|\tau)}{2\theta\left[\begin{smallmatrix} 1/2+i/n \\ 1/2-j/n \end{smallmatrix}\right](-2\hat{\Lambda}/n|\tau)\theta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](z - z_*|\tau)} . \end{aligned} \tag{3.82}$$

$\hat{\mu}_*, \hat{\Lambda} \in \mathbb{C}[[\hbar]]$ are related to the parameters q_* by

$$\hat{\mu}_* = \frac{2q_*}{\sum_{z_*} q_*} + \mathcal{O}(\hbar) , \qquad \hat{\Lambda} = -\frac{\hbar}{2} \sum_{z_*} q_* + \mathcal{O}(\hbar^2) . \tag{3.83}$$

On the right hand side of (3.81) is a finite K-matrix solving the BYBE for Belavin's symmetric R-matrix, appropriately interpreted as a quasi-classical K-matrix with classical limit τ_U. It originally appeared in [100].

Example 3.3.2. Now we turn our attention to n even. Let $n = 2m$. This is slightly less straightforward as the fixed point subalgebras $\mathfrak{p}_*$ are no longer all isomorphic. If $\xi_* \equiv \eta_* \equiv 0\,(2)$ then $\mathfrak{p}_* \cong \mathfrak{sl}_{m-1}(\mathbb{C}) \oplus \mathfrak{sl}_{m+1}(\mathbb{C}) \oplus \mathbb{C}$. Otherwise $\mathfrak{p}_* \cong \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_m(\mathbb{C}) \oplus \mathbb{C}$. Fortunately the generators of the abelian summands can easily expressed using the formula

$$Q_* = \frac{1}{m} \sum_{k,l \in \mathbb{Z}_n}{}' \, t_{2k - \zeta\eta_*, 2l - \zeta\xi_*} \epsilon^{-2\zeta^{-1} kl} . \tag{3.84}$$

Here the primed sum indicates that we remove the terms for which $(2k, 2l) \equiv (\zeta\eta_*, \zeta\xi_*)$. These only appear for $\xi_* \equiv \eta_* \equiv 0\,(2)$.

We take the bulk reflecting Wilson line to be in the fundamental representation V, and choose the four boundary Wilson lines to be in 1 dimensional representations with charges $Q_* \mapsto \tilde{q}_*$

The contribution of the bulk reflecting Wilson line is unchanged from (3.74), and from (3.12) the bulk-boundary contributions are

$$\sum_{z_*} \delta\ell_{*, U \otimes W_*}(z) = \frac{1}{2m} \sum_{z^*} \sum_{k,l \in \mathbb{Z}_m}{}' \, \tilde{q}_*' \epsilon^{-2\zeta^{-1} kl} T_{2k - \zeta\eta_*, 2l - \zeta\xi_*} w_{2k - \zeta\eta_*, 2l - \zeta\xi_*}(z - z^*) . \tag{3.85}$$

where the $\tilde{q}'_*$ are given by

$$\tilde{q}'_* = \frac{2m}{m^2 - \delta_{a_* \equiv 0\,(2)}\delta_{b_* \equiv 0\,(2)}} \epsilon^{\zeta^{-1}(\zeta\xi_* - a_*)((\zeta\eta_* - b_*)/2) - \zeta^{-1}a_* b_*} \tilde{q}_* \,. \tag{3.86}$$

Combining this with the self-interaction term (3.74) shows that the full classical ℓ-matrix is

$$\ell_{U\otimes W}(z) = \frac{1}{m}\sum_{z^*} q_* \sum_{k,l\in\mathbb{Z}_m}{}' \epsilon^{-2\zeta^{-1}kl} T_{2k-\zeta\eta_*, 2l-\zeta\xi_*} w_{2k-\zeta\eta_*, 2l-\zeta\xi_*}(z - z_*) \tag{3.87}$$

for

$$q_* = \frac{1}{2}\delta_{a_* \equiv 0\,(2)}\delta_{b_* \equiv 0\,(2)}\epsilon^{\zeta\xi_*(\eta_*/2)} + \frac{1}{2}\tilde{q}'_* \,. \tag{3.88}$$

In appendix A.5 we show that, for $\zeta = 1$ and $\xi = \eta = 0$, to first order in $\hbar$ this is consistent with

$$K(z) \sim \left(\sum_{z_*} \hat{\mu}_* \sum_{k,l\in\mathbb{Z}_m^2} L_*^{2k+a_*, 2l+b_*}(z, \hat{\Lambda}) T_{2k+a_*, 2l+b_*}\right)\tau_U \tag{3.89}$$

for

$$\begin{aligned} &L^{2k+a_*, 2l+b_*}(z; \hat{\Lambda}) \\ &\quad = \hat{\mu}_* \varepsilon^{-2kl} e^{\pi i b_*(2\hat{\Lambda} + a_*)/n} \frac{\theta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](-2\hat{\Lambda}/n|\tau)\,\theta\left[\begin{smallmatrix} 1/2+2k/n+a_*/n \\ 1/2-2l/n-b_*/n \end{smallmatrix}\right](z - z_* - 2\hat{\Lambda}/n|\tau)}{\theta\left[\begin{smallmatrix} 1/2+2k/n+a_*/n \\ 1/2-2l/n-b_*/n \end{smallmatrix}\right](-2\hat{\Lambda}/n|\tau)\,\theta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](z - z_*|\tau)} \,. \end{aligned} \tag{3.90}$$

$\hat{\mu}_*, \hat{\Lambda} \in \mathbb{C}[[\hbar]]$ are related to the parameters q_* by

$$\hat{\mu}_* = \frac{q_*}{q_*|_{a_*=b_*=0}} + \mathcal{O}(\hbar)\,, \qquad \hat{\Lambda} = -\hbar q_*|_{a_*=b_*=0} + \mathcal{O}(\hbar^2)\,. \tag{3.91}$$

On the right hand side of (3.89) is a finite K-matrix solving the BYBE for Belavin's symmetric R-matrix, appropriately interpreted as a quasi-classical K-matrix with classical limit τ_U. It also originally appeared in [100].

We have seen above that CS_4 generates elliptic K-matrices which agree to first order in $\hbar$ with those appearing in [100]. These depend on four complex parameters which correspond to the four charges on the boundary Wilson lines. We conjecture that this agreement holds (up to scale) to all orders in $\hbar$.

It is natural to ask if these exhaust all possible solutions to the BYBE for Belavin's symmetric R-matrix. Unfortunately they do not. In [72] K-matrices were obtained depending on $n + 1$ free complex parameters. These seem to have no obvious origin in the gauge theory description: they appear not to have a natural classical limit, and the points $z_* = (a_* + b_*\tau)/2$ do not play a distinguished role. Despite this, the K-matrices we do obtain conjecturally encompass a large class of solutions, including the generic case for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ [92, 56] and Cherednik's original example [39].

Remark. Much like previous examples in subsections 2.2.6, 3.2.1 & 3.2.3 the classical ℓ-matrices (3.80) & (3.87) are 'independent' of $\hbar$ in the sense introduced in 2.2.6. Even in the elliptic case we do not believe this behaviour to be generic, for example, if we take V to be the adjoint then the ℓ-matrix does not have this property.[3]

3.4 Discussion

In this chapter we have seen how to generate trigonometric and elliptic solutions to the BYBE using CS_4. We explicitly computed the order $\hbar$ contributions to K-matrices in some simple cases, and matched them to finite K-matrices appearing in the literature, appropriately interpreted as quasi-classical K-matrices.

One natural question we have not yet addressed is what the algebras playing the role of the twisted Yangian in the trigonometric and elliptic cases are. We expect boundary line operators in CS_4 to be labelled by representations of these algebras.

The analogue of the Yangian in the trigonometric case is the quantized loop algebra of $\mathfrak{g}_0$ [64]. In [51] the RTT presentation of this algebra was obtained from CS_4 in much the same way as it arises for the Yangian. It is closely related to the q-deformation of the untwisted affine algebra $\hat{\mathfrak{g}}_0$, denoted $U_q(\hat{\mathfrak{g}}_0)$ [35]. Heuristically the finite parameter q is related to $\hbar$ by $q = \exp \hbar$. It is instructive to think of the quantized loop algebra of $\mathfrak{g}_0$ as an infinitesimal counterpart of $U_q(\hat{\mathfrak{g}}_0)$.

The analogues of the twisted Yangian in the trigonometric case are coideal subalgebras of the quantized loop algebra of $\mathfrak{g}_0$. These are, in turn, infinitesimal counterparts of coideal subalgebras of $U_q(\hat{\mathfrak{g}}_0)$. Examples of the latter arise as q-deformations of $U(\hat{\mathfrak{g}}_0^\theta)$ for $\hat{\mathfrak{g}}_0^\theta$ the fixed point subalgebra of an involutive automorphism θ of $\hat{\mathfrak{g}}_0$.

The constant automorphisms σ used in examples 3.2.1 & 3.2.2, and more generally constructed in appendix A.2, can be viewed as a subset of the z-dependent automorphisms generated in appendix A.3. These are obtained starting with involutive automorphisms θ of $\hat{\mathfrak{g}}_0$. It is natural to expect that the algebras we seek are related to quantizations of $U(\hat{\mathfrak{g}}_0^\theta)$ in $U_q(\hat{\mathfrak{g}}_0)$.

[3] In [40] it is demonstrated that the adjoint representation of $\mathfrak{sl}_n(\mathbb{C})$ lifts to the elliptic quantum group, therein referred to as a generalised Sklyanin algebra.

For the constant automorphisms σ used in subsection 3.2.1 there are natural candidates for these quantizations. The choice made in example 3.2.1 is expected to correspond to the q-Onsager algebra [156, 16], and in example 3.2.2 is expected to correspond to the q-deformed twisted Yangian of type AIII [37]. We anticipate that the RTT presentations of these algebras, or rather their infinitesimal counterparts, can be recovered from CS_4 using the methods of [51]. (See also [128] for RTT presentations of the q-deformed twisted Yangians of type AI and AII in a form which can be easily compared that of the quantum loop algebra in [51]. These should arise in CS_4 from taking σ to be an outer automorphism of $\mathfrak{sl}_n(\mathbb{C})$.)

The z-dependent automorphism used in example 3.2.3 is predicted to relate to the coideal subalgebra introduced in proposition 2.1 of [147]. We do not believe the algebra corresponding to the z-dependent automorphism used in example 3.2.4 has been studied explicitly before. It would be interesting to obtain RTT presentations from CS_4 in both cases.

All of these quantizations are specialisations of a general class of quantum group analogues of $U(\hat{\mathfrak{g}}_0^\theta)$ constructed in [99] employing the methods of [103]. These are known as quantum symmetric pairs. We expect that general choices of the z-dependent automorphism $\hat{\sigma}(z)$ as identified in appendix A.3 will lead to infinitesimal counterparts of a subset of these algebras. It would be very interesting to establish this more concretely.

In the elliptic case the analogue of the Yangian is Belavin's elliptic quantum group. The counterparts of the twisted Yangian are presumably coideal subalgebras of this, although they do not appear to have been studied in the literature. Using CS_4 it is to be expected that one can define these coideal subalgebras through their categories of representations along the lines of [51, 69], though we do not attempt to do so here.

One major limitation of our approach to generating trigonometric and elliptic K-matrices using CS_4 is our inability to deduce all orders results for K-matrices. It is worth taking a moment to explain why the proof of uniqueness in rational case given in subsection 2.3.5 does not straightforwardly generalise. It proceeds by comparing two K-matrices which agree to order $\hbar^{r-1}$ for $r \geq 2$

$$K'(z) = K(z) + \hbar^r \delta\ell_r(z)\tau_U(z) + \mathcal{O}(\hbar^{r+1})\,. \tag{3.92}$$

Under the assumption that they both have unit Sklyanin determinant we learn that $\mathrm{tr}(\delta\ell_r(z)) = 0$. The Sklyanin determinant arises from CS_4 in exactly the same way in the trigonometric and elliptic cases as in the rational case, so this step is unchanged.

Substituting into the BYBE gives

$$\begin{aligned} 0 =&[\delta\ell_{r,1}(z_1), r_{12}(z_1 - z_2) - \hat{\sigma}_2(z_2) r_{12}(z_1 + z_2)] \\ &+ [\delta\ell_{r,2}(z_2), r_{12}(z_1 - z_2) + \hat{\sigma}_2(z_2) r_{12}(z_1 + z_2)] \end{aligned} \tag{3.93}$$

at order $\hbar^{r+1}$. This is a linearisation of the CBYBE for a z-dependent $\hat{\sigma}$. Using CS_4 we have generated affine spaces of solutions to the CBYBE, and the vector spaces on which these are modelled solve its linearisation. The corresponding failures in uniqueness can be accounted for by allowing the charges labelling boundary Wilson lines to take values in $\mathbb{C}[[\hbar]]$ instead of $\mathbb{C}$. Unfortunately, in the trigonometric and elliptic cases, we do not know if all solutions to equation (3.93) arise in this way.

For the YBE this issue was circumvented in [51] using the classification of solutions to the CYBE in [22], and in the rational case it was possible to solve equation (3.93) on account of the simplicity of the bulk rational r-matrix. It can probably be solved in general using the methods of [22], but we will not attempt to do so here.

Chapter 4

Twistor actions for 4d integrable theories

Twistors, introduced by Penrose in [141], are a powerful tool for describing the geometry and physics of 4d spacetime using complex methods. The simplest illustration of this is the Penrose transform, which expresses solutions to the zero rest mass equations on Minkowski space in terms of cohomology classes on twistor space [142, 143]. Remarkably, this correspondence between differential equations on spacetime and holomorphic structures on twistor space extends to certain non-linear cases. In particular, Penrose's nonlinear graviton construction relates anti-self-dual (ASD) metric perturbations to complex structure deformations [144], and the Ward correspondence links ASD connections to holomorphic G-bundles [164]. The latter was applied in Euclidean signature to the problem of constructing instantons [12, 11], eventually leading to a full classification [10]. The twistor description of these theories makes integrability manifest.

Interest in twistors was renewed by Witten's introduction of the twistor string [169]. He argued that the open B-model topological string on $\mathcal{N} = 4$ super twistor space, which has an effective target space description as HCS, was equivalent to ASD $\mathcal{N} = 4$ super Yang-Mills on spacetime. Furthermore, accounting for D-instanton contributions led to full $\mathcal{N} = 4$ super Yang-Mills. In the multitude of works that followed significant attention was directed at identifying twistor actions for spacetime theories, with considerable success. At this point twistor actions for Yang-Mills theory [121, 32], conformal gravity [28, 121] and Einstein gravity [123, 151] have all been obtained, along with their supersymmetric counterparts. To describe non-ASD fields it is necessary to introduce non-local terms on twistor space, which break integrability.

In this chapter we will take a rather different perspective. Instead of attempting to describe non-integrable theories on twistor space, we use local actions on twistor space to determine 4d analogues of 2d integrable theories. Inspired by a recent proposal of Costello [44], we study HCS on twistor space defined using a meromorphic, as opposed to holomorphic, $(3,0)$-form. We find that the resulting spacetime theories admit a kind of Lax connection. For the simplest choices of $(3,0)$-form, i.e., those which are nowhere vanishing, they have classical equations of motion equivalent to the ASDYM equations, a consequence of the Ward correspondence. Unfortunately the cost of integrability is to break Lorentz invariance.

We also discuss how to extend these actions to Lorentzian and ultrahyperbolic signatures, and how to impose suitable reality conditions in HCS so as to induce natural reality conditions on the corresponding 4d integrable theory.

This chapter is largely based on the paper [31]. (It has some overlap with [140], which appeared on the arXiv at around the same time.)

Throughout this chapter we use spinor notation and homogeneous coordinates on projective spaces. These are reviewed in appendices B.3 & B.4 respectively. Note that some notation employed in this and the following chapter differs from that used in chapters 2 & 3.

4.1 Euclidean twistors

We begin by reviewing Euclidean twistors, as introduced in [11].

The twistor space (sometimes referred to as the projective twistor space) of complexified spacetime, $\mathbb{CM}^4 = \mathbb{C}^4$, is $\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$. (Note that Penrose instead takes $\mathbb{PT} = \mathbb{CP}^3$, and writes $\mathbb{PT}' = \mathbb{CP}^3 \setminus \mathbb{CP}^1$.) $\mathbb{CP}^3$ may be provided with homogeneous coordinates $Z^\alpha = (\omega^A, \pi_{A'})$ defined with respect the equivalence relation $Z^\alpha \sim tZ^\alpha$ for $t \in \mathbb{C}^*$. It is convenient to remove the $\mathbb{CP}^1$ with coordinates $(\omega^A, 0)$. As a complex manifold $\mathbb{PT}$ is biholomorphic to the total space of the line bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$, the coordinates $(\omega^A, \pi_{A'})$ parametrise the fibres and base respectively. Subspaces of particular interest are holomorphic lines $\iota_x : \mathbb{CP}^1_x \hookrightarrow \mathbb{PT}$, $\pi_{A'} \mapsto (x^{AB'}\pi_{B'}, \pi_{A'})$, parametrised by $x \in \mathbb{CM}^4$.

Complex conjugation in Euclidean signature preserves the handedness of spinors, acting by

$$\omega^A \mapsto \hat{\omega}^A = (-\overline{\omega^1}, \overline{\omega^0}) , \quad \pi^{A'} \mapsto \hat{\pi}^{A'} = (-\overline{\pi^{1'}}, \overline{\pi^{0'}}) . \tag{4.1}$$

It has no fixed points, even when viewed as acting on projective spinors, and induces SU_2 invariant inner products $\|\omega\|^2 = \omega^A \hat{\omega}_A$ and $\|\pi\|^2 = \pi^{A'} \hat{\pi}_{A'}$. We can extend spinor conjugation to act on $\mathbb{PT}$ in the obvious way

$$Z^\alpha = (\omega^A, \pi_{A'}) \mapsto \hat{Z}^\alpha = (\hat{\omega}^A, \hat{\pi}_{A'})\,, \tag{4.2}$$

and this also has no fixed points. It provides $\mathbb{PT}$ with a non-holomorphic fibration over $\mathbb{R}^4$, given by

$$Z^\alpha = (\omega^A, \pi_{A'}) \mapsto x^{AA'} = \frac{\hat{\omega}^A \pi^{A'} - \omega^A \hat{\pi}^{A'}}{\|\pi\|^2}\,. \tag{4.3}$$

The image is real, and $\varepsilon_{AB}\varepsilon_{A'B'}x^{AA'}x^{BB'} = \delta_{ab}x^a x^b = x^2$ for δ the standard Euclidean metric on $\mathbb{R}^4$. The fibre over $x \in \mathbb{R}^4$ is $\mathbb{CP}^1_x$. This fibration provides a smooth identification between $\mathbb{PT}$ and the left-handed projective spinor bundle over $\mathbb{R}^4$, $\mathbb{PS}^+ \cong \mathbb{CP}^1 \times \mathbb{R}^4$. Explicitly

$$\mathbb{PT} \cong \mathbb{PS}^+\,, \qquad (\omega^A, \pi_{A'}) \mapsto (x^{AA'}, \pi_{A'})\,. \tag{4.4}$$

We continue to use the inhomogeneous coordinates $\pi_{A'}$ along the fibres.

It will prove useful to introduce the following frame of $(0,1)$-forms adapted to the non-holomorphic coordinates $(x^{AA'}, \pi_{A'})$, originally introduced in [32]

$$\bar{e}^0 = \frac{\langle \mathrm{d}\hat{\pi}\,\hat{\pi}\rangle}{\|\pi\|^4}\,, \qquad \bar{e}^A = \frac{\mathrm{d}x^{AA'}\hat{\pi}_{A'}}{\|\pi\|^2}\,. \tag{4.5}$$

Note that $\bar{e}^0$ and $\bar{e}^A$ have holomorphic weight -2 and -1 respectively, and have vanishing antiholomorphic weight. They are holomorphic, but $\bar{e}^A$ is not closed. The corresponding dual frame of vector fields with values in the antiholomorphic tangent bundle is

$$\bar{\partial}_0 = \|\pi\|^2 \pi^{A'} \frac{\partial}{\partial \hat{\pi}^{A'}}\,, \qquad \bar{\partial}_A = \pi^{A'}\partial_{AA'}\,. \tag{4.6}$$

It is also useful to introduce an analogous frame of $(0,1)$-forms given by

$$e^0 = \langle \mathrm{d}\pi\,\pi\rangle\,, \qquad e^A = \mathrm{d}x^{AA'}\pi_{A'}\,. \tag{4.7}$$

Note that it does not merely consist of the conjugates of the frame (4.5). e^0 and e^A have holomorphic weight 2 and 1 respectively, and are antiholomorphic but the latter is not closed. By choosing our frames as above we ensure that the components of differential forms are sections of holomorphic bundles, which carry natural actions of the $\bar{\partial}$ operator.

There is a canonical holomorphic 3-form on $\mathbb{PT}$ with holomorphic weight 4, given by

$$\mathrm{D}^3Z = \frac{\varepsilon_{\alpha\beta\gamma\delta}Z^\alpha\,\mathrm{d}Z^\beta\wedge\mathrm{d}Z^\gamma\wedge\mathrm{d}Z^\delta}{4!} = \frac{e^0\wedge e^A\wedge e_A}{2} = \frac{\langle\mathrm{d}\pi\,\pi\rangle\wedge\mathrm{d}^2x^{A'B'}\pi_{A'}\pi_{B'}}{2}\,, \quad (4.8)$$

where we have defined $\mathrm{d}^2x^{A'B'} = \mathrm{d}x^{AA'}\wedge\mathrm{d}x^{BB'}\varepsilon_{AB}$.

4.2 HCS on twistor space

HCS was first introduced by Witten in [168]. The dynamical field is a partial connection, $\bar\partial+\bar{\mathcal{A}}$, on a principal G-bundle over a complex 3-manifold X with structure group a reductive complex group G. The action takes the standard form

$$S_\Omega[\bar{\mathcal{A}}] = \frac{1}{2\pi i}\int_X \Omega\wedge\mathrm{HCS}(\bar{\mathcal{A}})\,, \quad (4.9)$$

where

$$\mathrm{HCS}(\bar{\mathcal{A}}) = \mathrm{tr}\bigg(\bar{\mathcal{A}}\wedge\bar\partial\bar{\mathcal{A}}+\frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\bigg)\,. \quad (4.10)$$

Here tr is a $\mathfrak{g}$-invariant bilinear which for $\mathfrak{g}$ simple we take to be the minimal bilinear introduced in section 1.3.2, and Ω is a $(3,0)$-form. The action is invariant under infinitesimal gauge transformations of the form

$$\delta\bar{\mathcal{A}} = \bar\partial\varepsilon+[\bar{\mathcal{A}},\varepsilon] \quad (4.11)$$

as long as Ω is holomorphic. Together with the restrictive assumption of compactness, this implies that X is Calabi-Yau.

The classical equations of motion of HCS imply that the partial connection satisfies the Cauchy-Riemann condition

$$(\bar\partial+\bar{\mathcal{A}})^2 = \bar\partial\bar{\mathcal{A}}+\bar{\mathcal{A}}\wedge\bar{\mathcal{A}} = \bar{\mathcal{F}}(\bar{\mathcal{A}}) = 0\,, \quad (4.12)$$

and so defines a holomorphic structure on the G-bundle. Modulo gauge a classical solution of HCS defines a holomorphic G-bundle up to isomorphism.

The requirement that Ω be holomorphic precludes us taking $X=\mathbb{CP}^3$, the compact twistor space of S^4, since it is not Calabi-Yau. Worse still, it excludes $X=\mathbb{PT}=\mathcal{O}(1)\oplus\mathcal{O}(1)\to\mathbb{CP}$, the twistor space of Euclidean $\mathbb{R}^4$. This can be seen by noting that any $(3,0)$-form on $\mathbb{PT}$ can be expressed as $\mathrm{D}^3Z\otimes\Phi$ for Φ a smooth section of $\mathcal{O}(-4) \to \mathbb{PT}$. Taking Ω to be of this form, holomorphicity implies that Φ be holomorphic, or equivalently a meromorphic section of $\mathcal{O}(-4) \to \mathbb{CP}^3$ with poles on the support of a $\mathbb{CP}^1$ subspace. No suitable sections exist.

In his seminal paper [169] Witten circumvented this issue by working studying HCS on the super Calabi-Yau manifold $\mathbb{CP}^{3|4}$. It describes ASD $\mathcal{N} = 4$ super Yang-Mills on spacetime. (Notably it can be deformed to full $\mathcal{N} = 4$ super Yang-Mills by adding a non-local term [32].) By truncating the superfield expansion one obtains holomorphic BF theory on twistor space, which descends to a BF theory on spacetime which imposes the ASDYM equations with a Lagrange multiplier [169, 121]. Whilst this action is certainly useful in the study of scattering amplitudes, it is less interesting from the perspective of integrable systems.

Recently Costello [44] has proposed an alternative resolution. Instead of using a holomorphic Ω, he suggests using the meromorphic measure

$$\Omega = \frac{\mathrm{D}^3 Z}{(A \cdot Z)^2 (B \cdot Z)^2} \tag{4.13}$$

on $X = \mathbb{CP}^3$ for $A, B \in (\mathbb{C}^4)^\vee$ dual twistors. This measure has double poles on the loci $A \cdot Z = 0$ and $B \cdot Z = 0$, which define $\mathbb{CP}^2$ subspaces intersecting on a $\mathbb{CP}^1$. Excising this $\mathbb{CP}^1$ from $\mathbb{CP}^3$ gives $\mathbb{PT}$, and in the standard $Z = (\omega, \pi)$ coordinates $A = (0, \alpha)$, $B = (0, \beta)$ for left handed spinors α, β. In these coordinates

$$\Omega = \frac{e^0 \wedge e^A \wedge e_A}{2\langle \pi\, \alpha \rangle^2 \langle \pi\, \beta \rangle^2}\,. \tag{4.14}$$

Since Ω is no longer holomorphic, boundary terms are generated when varying the action. These must be eliminated by imposing suitable boundary conditions on the gauge field. Furthermore, the explicit dependence of the action on the spinors α, β breaks Lorentz invariance. Costello claims that for this choice of Ω HCS describes a 4d analogue of WZW_2 on spacetime. We shall verify this explicitly in section 4.3.1.

Inspired by this proposal, we consider a range of different meromorphic measures on $\mathbb{PT}$ generalising Costello's choice. We take

$$\Omega = \mathrm{D}^3 Z \otimes \Phi \tag{4.15}$$

for Φ a meromorphic section of $\mathcal{O}(-4) \to \mathbb{PT}$. In this text we will further assume that Φ factors through $\mathbb{PT} = \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}$, i.e., that it only depends on the π

coordinate on $\mathbb{PT}$.[1] This restriction is motivated by the observation that it will lead to effective spacetime actions which are independent of the spacetime coordinates and therefore invariant under translations. We can view Φ as the 4d analogue of the twist function in 2d CIFT [158]. Varying the action on the support of the classical equations of motion gives

$$\delta S_\Omega = \frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \bar\partial(\mathrm{tr}(\delta\bar{\mathcal{A}}\wedge\bar{\mathcal{A}})) = \frac{1}{2\pi i}\int_{\mathbb{PT}} \bar\partial\Omega \wedge \mathrm{tr}(\delta\bar{\mathcal{A}}\wedge\bar{\mathcal{A}})\,, \tag{4.16}$$

and so we will have to impose boundary conditions on $\bar{\mathcal{A}}$ to eliminate terms generated at the poles of Ω. On the other hand, if Ω has zeros we may be able to tolerate poles in $\bar{\mathcal{A}}$ without introducing such terms.

To obtain the effective spacetime description of HCS theory we will always adopt the same basic approach employed in [53] and reviewed in subsection 1.3.6 to relate CS_4 to 2d CIFTs.

The first step is to fix the gauge on the fibres of $\mathbb{PT} \xrightarrow{\pi} \mathbb{R}^4$ by writing

$$\iota_x^*\bar{\mathcal{A}} = \hat\sigma^{-1}\bar\partial_{\mathbb{CP}^1_x}\hat\sigma \tag{4.17}$$

for an appropriate choice of frame field $\hat\sigma : \mathbb{PT} \to G$. By doing this we are assuming that when restricted to $\mathbb{CP}^1_x$ the complex structure $\bar\partial_{\mathbb{CP}^1_x} + \iota_x^*\bar{\mathcal{A}}$ is holomorphically trivial, a standard assumption of the Ward correspondence. A finite gauge transformation with parameter $g : \mathbb{PT} \to G$ acts on $\hat\sigma$ by

$$\hat\sigma \mapsto \hat\sigma g^{-1}\,. \tag{4.18}$$

Compatibility with the boundary conditions on $\bar{\mathcal{A}}$ will impose restrictions on g, and so prevent us from setting $\hat\sigma = \mathrm{id}$.. Nonetheless, we will find that $\hat\sigma$ can always be expressed in terms of a finite number of group or Lie algebra valued fields on $\mathbb{R}^4$.

There is a second redundancy in the gauge fixing (4.17) under

$$\hat\sigma \mapsto h\hat\sigma \tag{4.19}$$

for $h : \mathbb{R}^4 \to G$. This is not some residual gauge freedom on $\mathbb{PT}$ that survives gauge fixing, but rather a redundancy in our parametrisation of gauge equivalence classes using $\hat\sigma$. Often it is convenient to eliminate this ambiguity, e.g., by setting $\hat\sigma|_{\pi\sim\pi_0} = \mathrm{id}$.

[1] In many cases, like that considered by Costello, it's possible to exploit conformal symmetry to bring Ω into a form in which it is independent of the coordinates ω. More generally, it is natural to choose Φ on the basis of what subgroup of the conformal group it leaves invariant.

at some π_0. We will see shortly that this can naturally be interpreted as a gauge symmetry on spacetime. This same redundancy was observed for CS_4 in [60].

It is then natural to write

$$\bar{\partial} + \bar{\mathcal{A}} = \hat{\sigma}^{-1}(\bar{\partial} + \bar{\mathcal{A}}')\hat{\sigma} \tag{4.20}$$

for some $\bar{\mathcal{A}}'$ obeying $\iota_x^* \bar{\mathcal{A}}' = 0$, or in components $\bar{\mathcal{A}}' = \bar{e}^A \bar{\mathcal{A}}'_A$. We should think of $\bar{\mathcal{A}}'$ as being $\bar{\mathcal{A}}$ in a 'gauge' in which $\bar{\mathcal{A}}_x$ vanishes. Under the redundancy $\hat{\sigma} \mapsto h\hat{\sigma}$ we find that

$$\bar{\mathcal{A}}'_A \mapsto -h^{-1}\bar{\partial}_A h + h\bar{\mathcal{A}}'_A h^{-1} \,. \tag{4.21}$$

Next we partially impose the classical equations of motion. We can split $\bar{\mathcal{F}}(\bar{\mathcal{A}})$ into components which are normal to the holomorphic curves $\mathbb{CP}_x$, and those which are not. With respect to the frame $\bar{e}^0, \bar{e}^A$ introduced in equation 4.5 these are given by

$$\frac{1}{2}\varepsilon^{AB}\bar{\mathcal{F}}(\bar{\mathcal{A}})_{BA} = \bar{\partial}^A \bar{\mathcal{A}}_A + \frac{1}{2}[\bar{\mathcal{A}}^A, \bar{\mathcal{A}}_A] \,, \tag{4.22}$$

$$\bar{\mathcal{F}}(\bar{\mathcal{A}})_{0A} = \bar{\partial}_0 \bar{\mathcal{A}}_A - \bar{\partial}_A \bar{\mathcal{A}}_0 + [\bar{\mathcal{A}}_0, \bar{\mathcal{A}}_A] \,. \tag{4.23}$$

To descend to spacetime we impose the latter. In terms of $\bar{\mathcal{A}}'$ it linearises to

$$\bar{\partial}_0 \bar{\mathcal{A}}'_A = 0 \tag{4.24}$$

and so we learn that $\bar{\mathcal{A}}'$ is holomorphic up the twistor fibres. The boundary conditions on $\bar{\mathcal{A}}$ then fix $\bar{\mathcal{A}}'$ in terms of $\hat{\sigma}$.

At this point the dependence of $\bar{\mathcal{A}}$ on π is completely determined, and we can integrate over the twistor fibres to get a 4d spacetime action. Its classical equations of motion will be equivalent to equation (4.22). There are two natural ways to interpret this:

- If we choose Φ so that it's nowhere vanishing, then $\bar{\mathcal{A}}$ will be without poles.[2] Then for $\bar{\mathcal{A}}'_A$ to be holomorphic we must have

$$\bar{\mathcal{A}}' = \bar{e}^A \bar{\mathcal{A}}'_A = \bar{e}^A \pi^{A'} A_{AA'} \,, \tag{4.25}$$

[2]This is not necessarily true: we can tolerate poles in $\bar{\mathcal{A}}$ at points where Φ is non-vanishing as long as the residues are carefully chosen so that the boundary terms generated when varying the action disappear. In fact symmetry reductions of HCS theory of the type discussed in chapter 5 can lead to analogous boundary conditions in CS_4, though we will not consider them in this text.

with $A_{AA'}$ independent of π. In terms of $\nabla_{AA'} = \partial_{AA'} + A_{AA'}$ the equations of motion of the effective spacetime theory are

$$\bar{\mathcal{F}}_{AB}(\bar{\mathcal{A}}') = \pi^{A'}\pi^{B'}[\nabla_{AA'}, \nabla_{BB'}] = 0\,, \tag{4.26}$$

which we recognise as the ASDYM equations. (See appendix B.5 for background on the ASDYM equations.) This is precisely what we expected from the Penrose-Ward transformation. From equation (4.21), under the redundancy $\hat{\sigma} \mapsto h\hat{\sigma}$ we find that

$$A_{AA'} \mapsto -h^{-1}\partial_{AA'}h + hA_{AA'}h^{-1}\,, \tag{4.27}$$

and so it is indeed natural to interpret A as a connection on spacetime. The boundary conditions on $\bar{\mathcal{A}}$ place restrictions on A which can naturally be interpreted as gauge fixing conditions. The equations of motion we obtain from the effective action on $\mathbb{R}^4$ will be equivalent to the ASDYM equations in this gauge.

- Alternatively we can view $\mathcal{L}_A = \bar{\mathcal{A}}'_A$ as a 4d analogue of the Lax connection of a 2d integrable system. Then

$$\bar{\partial}^A\mathscr{L}_A + \frac{1}{2}[\mathscr{L}^A, \mathscr{L}_A] = 0 \tag{4.28}$$

is the zero-curvature equation. This interpretation has the advantage of extending to the case where Φ has zeros and $\bar{\mathcal{A}}$ has poles. We will see in chapter 5 that under symmetry reduction $\mathscr{L}_A$ descends to the Lax connection of a 2d CIFT.

Remark. It is instructive to attempt to employ the above argument in the quantum setting. Gauge fixing in exactly the same way introduces a Faddeev-Popov determinant

$$\det\left(\bar{\partial}_{\mathbb{CP}^1_x} + \iota_x^*\bar{\mathcal{A}}\right) \tag{4.29}$$

inside the path integral. Instead of imposing the equation of motion (4.23) by hand, we instead integrate out the components $\bar{A}_A$. This introduces a compensating determinant

$$\det\left(\bar{\partial}_{\mathbb{CP}^1_x} + \iota_x^*\bar{\mathcal{A}}\right)^{-1}. \tag{4.30}$$

At first glance these look as though they may cancel, but this is too naive. In the former determinant $\bar{\partial}_{\mathbb{CP}^1_x} + \iota^*\bar{\mathcal{A}}$ acts on infinitesimal gauge transformations, whereas in the latter it acts on the components $\bar{\mathcal{A}}_A$. These carry holomorphic weights 0 and 1 respectively and in general obey different boundary conditions. Ignoring possible contributions from these determinants, the resulting effective action on 4d spacetime agrees with the one derived classically.

We expect that this discrepancy in the determinants is related to the observation in [48] that HCS suffers from a 1-loop anomaly.

4.3 Examples of twistor actions for the ASDYM equations

In this section we work through the construction described above for a number of different choices of the $(3,0)$-form Ω and corresponding boundary conditions on $\bar{\mathcal{A}}$. In all of these examples we will take Ω to be nowhere vanishing, which we find leads to spacetime actions with classical equations of motion equivalent to the ASDYM equations. Further exotic examples can be found in the appendix of [31].

4.3.1 WZW_4

We begin by verifying Costello's assertion in [44] that for

$$\Omega = \frac{\mathrm{D}^3 Z}{(A\cdot Z)^2(B\cdot Z)^2} = \frac{e^0\wedge e^A\wedge e_A}{2\langle\pi\,\alpha\rangle^2\langle\pi\,\beta\rangle^2} \tag{4.31}$$

HCS describes WZW_4 on $\mathbb{R}^4$. (Recall that $A, B\in(\mathbb{C}^4)^\vee$ are dual twistors represented by $(0,\alpha)$ and $(0,\beta)$.) See [140] for an alternative proof. The first step towards doing so is to identify suitable boundary conditions on $\bar{\mathcal{A}}$ which eliminate the boundary terms which arise when varying the action. The obvious choice is to require that $\bar{\mathcal{A}}$ vanish to first order at $\pi\sim\alpha,\beta$, or equivalently that $\bar{\mathcal{A}}$ is divisible by $\langle\pi\,\alpha\rangle\langle\pi\,\beta\rangle$ and can be expressed in the form

$$\bar{\mathcal{A}} = \langle\pi\,\alpha\rangle\langle\pi\,\beta\rangle\Phi \tag{4.32}$$

for $\Phi\in\Omega^{0,1}(\mathbb{PT},\mathfrak{g}\otimes\mathcal{O}(-2))$. Gauge transformations similarly are required to vanish to first order at $\pi\sim\alpha,\beta$.

This example bears a striking similarity to the CS_4 description of the PCM reviewed in section 1.3.6. We will this explore connection in greater detail in chapter 5.

To determine the effective spacetime action we follow the procedure outlined in the previous section. We begin by writing

$$\iota_x^*\bar{\mathcal{A}} = \hat{\sigma}^{-1}\bar{\partial}_{\mathbb{CP}^1_x}\hat{\sigma} \tag{4.33}$$

for some $\hat{\sigma} : \mathbb{PT} \to G$. Under a finite gauge transformation with parameter g, which must obey $g|_{\pi\sim\alpha} = g|_{\pi\sim\beta} = \text{id.}$, $\hat{\sigma} \mapsto \hat{\sigma} g^{-1}$. This shows that $\hat{\sigma}$ is essentially arbitrary except for its values at $\pi \sim \alpha, \beta$. Using the right action $\hat{\sigma} \mapsto h\hat{\sigma}$ we can set $\hat{\sigma}|_{\pi\sim\beta} = \text{id.}$., leaving $\hat{\sigma}|_{\pi\sim\alpha} = \sigma$ as the gauge invariant data which can be extracted from $\iota_x^* \bar{\mathcal{A}}$. We can also interpret this as the holomorphic Wilson line from α to β

$$\mathcal{W}_{\alpha\to\beta} = \mathrm{P}\exp\left(-\frac{1}{2\pi i}\int_{\mathbb{CP}^1_x} \frac{\langle \alpha\, \beta\rangle\langle \mathrm{d}\pi\, \pi\rangle}{\langle \alpha\, \pi\rangle\langle \pi\, \beta\rangle} \wedge \iota_x^* A\right) = \hat{\sigma}^{-1}|_{\pi\sim\beta}\hat{\sigma}|_{\pi\sim\alpha} = \sigma\,. \tag{4.34}$$

Here a holomorphic Wilson line [120, 4, 3] is defined by

$$(\bar{\partial}_{\mathbb{CP}^1_x} + \iota_x^*\bar{\mathcal{A}})\mathcal{W}_{\pi_0\to\pi} = 0\,, \qquad \mathcal{W}_{\pi_0\to\pi_0} = \text{id.}\,. \tag{4.35}$$

Apart from its values at $\pi \sim \alpha, \beta$, the precise choice of $\hat{\sigma}$ is arbitrary. It's convenient to demand that it be equal to σ and id. in neighbourhoods of $\pi \sim \alpha$ and β respectively, and also invariant under the U(1) action on $\mathbb{CP}^1_x$ preserving α, β.[3]

We then simplify the equation of motion up the fibres (4.23) by defining $\bar{\mathcal{A}}'$ as in (4.20), in terms of which they read $\bar{\partial}_0 \bar{\mathcal{A}}'_A = 0$. The solution is

$$\bar{\mathcal{A}}'_A = \pi^{A'} A_{AA'} \tag{4.36}$$

with A independent of π. The boundary conditions on $\bar{\mathcal{A}}$ then imply that

$$\begin{aligned} \bar{\mathcal{A}}|_{\pi\sim\alpha} &= \alpha^{A'}(\sigma^{-1}\partial_{AA'}\sigma + \sigma^{-1}A_{AA'}\sigma) = 0\,,\\ \bar{\mathcal{A}}|_{\pi\sim\beta} &= \beta^{A'}A_{AA'} = 0\,. \end{aligned} \tag{4.37}$$

The second equation is solved by writing $A_{AA'} \sim \beta_{A'}$, and substituting this into the former gives

$$A_{AA'} = -\beta_{A'}\alpha^{B'}\partial_{AB'}\sigma\sigma^{-1}\,. \tag{4.38}$$

Here we are without loss of generality assuming that $\langle \alpha\, \beta\rangle = 1$. This is the standard expression for an ASD gauge field in terms of Yang's matrix σ. (See appendix B.5 for details.) It can be interpreted as a gauge fixing condition for an ASD connection. Note, however, that it is not attainable for a generic connection.

[3] In section 4.3.2 we make an alternative choice of gauge which is equally applicable here. We first write $\iota_x^*\bar{\mathcal{A}} = \langle \pi\, \alpha\rangle\langle \pi\, \beta\rangle\psi$ for ψ of holomorphic weight -2, and then require that ψ be harmonic with respect to the Fubini-Study metric on $\mathbb{CP}^1_x$. (In Euclidean signature spinor conjugation induces the Fubini-Study metric on $\mathbb{CP}^1_x$.) Following [172] we therefore have $\iota_x^*\bar{\mathcal{A}} = \varphi\langle \pi\, \alpha\rangle\langle \pi\, \beta\rangle\langle \mathrm{d}\hat{\pi}\, \hat{\pi}\rangle/\|\pi\|^4$ for φ a $\mathfrak{g}$-valued spacetime field. It is straightforward to verify that $\sigma = \exp\varphi$.

Now that the π dependence of $\bar{\mathcal{A}}$ is completely determined we can directly perform the integrals over the twistor fibres to reduce HCS to an effective action on $\mathbb{R}^4$. Similar calculations have appeared in section 1.3.6 and will be performed in the sequel so we omit the details here.

One finds that the resulting 4 dimensional action is

$$S_{\mathrm{WZW}_4}[\sigma] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}(J \wedge *J) + \frac{1}{3}\int_{\mathbb{R}^4\times[0,1]} \omega_{\alpha,\beta} \wedge \mathrm{tr}(\tilde{J}^3)\,, \tag{4.39}$$

where $J = -\,\mathrm{d}\sigma\sigma^{-1}$. Similarly, $\tilde{J} = -\tilde{\mathrm{d}}\tilde{\sigma}\tilde{\sigma}^{-1}$ for $\tilde{\sigma}$ a smooth homotopy from σ to id., with $\tilde{\mathrm{d}}$ the exterior derivative on $\mathbb{R}^4 \times [0,1]$. We have also introduced $\omega_{\alpha,\beta} = \mathrm{d}^2x^{A'B'}\alpha_{A'}\beta_{B'}$.

This action was originally proposed by Donaldson [66], and examined in detail in [109, 129]. We refer to it as the 4d WZW model (WZW_4). Varying with respect to σ gives

$$\varepsilon^{AB}\alpha^{A'}\beta^{B'}\partial_{AA'}(\sigma^{-1}\partial_{BB'}\sigma) = 0\,, \tag{4.40}$$

which we recognise as Yang's equation. This is consistent with our expectation that HCS theory on twistor space for this choice of Ω should be equivalent to the ASDYM equations on $\mathbb{R}^4$.

If we fix $\beta = \hat{\alpha}$, then $\omega_{\alpha,\beta} = \omega_{\alpha,\hat{\alpha}}$ is proportional to the Kähler form on $\mathbb{R}^4$ in the complex structure determined by α. At this point we are not obliged to make this choice of β, but we will see in section 4.5 that it will be necessary if we wish to obtain actions for ASD connections taking values in compact real forms of G. Let us assume that we do make this choice.

A number of properties of WZW_4 are manifest from the twistor perspective:

- The choice of spinor α and its conjugate $\hat{\alpha}$ break the left-handed SU(2) rotation symmetry to U(1), however the right-handed SU(2) is unbroken, leaving an $\mathrm{U}(1)\times\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{U}(2)$ subgroup of the rotation group SO(4). This is the group of symmetries compatible with a Kähler structure on $\mathbb{R}^4$. (Note that the conformal symmetry on twistor space is broken to the Ponicaré group by the infinity twistor $I^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma\delta}A_\alpha B_\beta$, so WZW_4 is not even classically a CFT.)

- For this choice of Ω, HCS is invariant under 'forbidden' gauge transformations with parameter $g : \mathbb{PT} \to G$ which do not vanish to first order at $\pi \sim \alpha, \hat{\alpha}$, but instead obey

$$\alpha^{A'}\partial_{AA'}g|_{\pi\sim\alpha} = 0\,, \qquad \hat{\alpha}^{A'}\partial_{AA'}g|_{\pi\sim\hat{\alpha}} = 0\,. \tag{4.41}$$

We may interpret these conditions as saying that $g|_{\pi\sim\alpha}$ is holomorphic and $g|_{\pi\sim\hat{\alpha}}$ is antiholomorphic with respect to the complex structure induced on $\mathbb{R}^4$ by α. Under this transformation

$$\sigma \mapsto g|_{\pi\sim\hat{\alpha}}\sigma g^{-1}|_{\pi\sim\alpha}\,, \tag{4.42}$$

which it straightforward to see is a symmetry of the action (4.39). These are 4d analogues of the familiar loop group symmetries of the WZW_2 model. By taking $g|_{\pi\sim\alpha}, g|_{\pi\sim\hat{\alpha}}$ to be constant we recover the right and left global G symmetries.

- As reviewed in section 1.3.6, CS_4 with $\omega = \mathrm{d}z/z$ describes WZW_2. For this ω, CS_4 is invariant under the U(1) action $z \mapsto e^{i\theta}z$. It is argued in [53] that by integrating over the phase of z one obtains the standard CS_3 description of WZW_2. In much the same way, in HCS we can integrate over the orbits of the natural U(1) action on $\mathbb{CP}^1$ preserving $\alpha, \hat{\alpha}$. This leads to the Kähler CS theory of Nair and Schiff [130, 129, 131].

Remark. It is interesting to note that WZW_4 arises as the target space description of the open and heterotic $\mathcal{N} = 2$ string [137, 138, 118], which has been conjectured to have a twistor origin [133, 119]. HCS is the target space description of the open B-model topological string [168] hinting at a possible connection between the two.

This concludes the overlap between the present work an the content of [44].

4.3.2 LMP action

Fix $A \in (\mathbb{C}^4)^\vee$ a dual twistor, and choose holomorphic coordinates (ω, π) on $\mathbb{C}^4$ with the respect to which it is represented by $(0, \alpha)$ for α a left handed spinor. Excising $\pi = 0$ and projectivising gives the twistor space $\mathbb{PT}$ of $\mathbb{R}^4$.

We show that for

$$\Omega = \frac{\mathrm{D}^3 Z}{(A\cdot Z)^4} = \frac{e^0 \wedge e^A \wedge e_A}{2\langle \pi\, \alpha\rangle^4} \tag{4.43}$$

HCS descends to an action originally proposed by Leznov and Mukhtarov, and Parkes [105, 104, 139]. As with WZW_4, its classical equations of motion are equivalent to the ADSYM equations. We refer to it as the LMP action.

To eliminate the boundary terms generated at the fourth order pole we must choose boundary conditions on $\bar{\mathcal{A}}$ so that

$$\mathrm{tr}(\delta\bar{\mathcal{A}} \wedge \bar{\mathcal{A}}) \tag{4.44}$$

vanishes to fourth order at $\pi \sim \alpha$. An obvious choice is to require that $\bar{\mathcal{A}}$ vanishes to second order, or equivalently that $\bar{\mathcal{A}}$ is divisible by $\langle\pi\,\alpha\rangle^2$. Infinitesimal gauge transformations are also required to vanish to second order at $\pi \sim \alpha$ to preserve this boundary condition.

To determine the effective spacetime action we follow the procedure outlined in section 4.2, although here we use a different method to fix the gauge than that adopted in [53].

We begin by pulling back $\bar{\mathcal{A}}$ to a twistor fibre in the usual way. Our boundary conditions guarantee that we can write $\iota_x^*\bar{\mathcal{A}} = \langle\pi\,\alpha\rangle^2\psi$ for ψ an element of $\Omega^{0,1}(\mathbb{CP}^1_x, \mathfrak{g}\otimes\mathcal{O}(-2))$. We then fix the gauge by requiring that ψ be harmonic with respect to the Fubini-Study metric on $\mathbb{CP}^1_x$. This closely mirrors the gauge condition adopted in [32]. Since $H^1(\mathbb{CP}^1_x, \mathcal{O}(-2)) \cong \mathbb{C}$, $\iota_x^*\bar{\mathcal{A}}$ is described by a single $\mathfrak{g}$-valued field $\phi : \mathbb{R}^4 \to \mathfrak{g}$. Explicitly $\psi = -\bar{e}^0\phi$, and so

$$\iota_x^*\bar{\mathcal{A}} = -\langle\pi\,\alpha\rangle^2\bar{e}^0\phi\,. \tag{4.45}$$

Introducing $\hat{\sigma}$ in the usual way and fixing the redundancy $\hat{\sigma} \to h\hat{\sigma}$ by setting $\hat{\sigma}|_{\pi\sim\alpha} = \text{id.}$, we find that

$$\hat{\sigma} = \exp\bigg(-\frac{\langle\pi\,\alpha\rangle\langle\hat{\pi}\,\alpha\rangle}{\|\pi\|^2}\phi\bigg)\,. \tag{4.46}$$

Conversely, the field ϕ may be extracted from $\hat{\sigma}$ as $\phi = \langle\pi\,\hat{\alpha}\rangle^{-2}\hat{\sigma}^{-1}\partial_0\hat{\sigma}|_{\pi\sim\alpha}$. It exhausts the gauge invariant data which can be extracted from $\iota_x^*\bar{\mathcal{A}}$.

After introducing $\bar{\mathcal{A}}'$ as in (4.20) and solving the equations of motion (4.23) by writing

$$\bar{\mathcal{A}}'_A = \pi^{A'}A_{AA'} \tag{4.47}$$

for A independent of π, the next step is to fix A in terms of ϕ using the boundary conditions. We begin by noting that

$$\begin{aligned} -\,\mathrm{d}\hat{\sigma}\hat{\sigma}^{-1} &= \frac{\langle\pi\,\alpha\rangle\langle\hat{\pi}\,\alpha\rangle}{\|\pi\|^2}\,\mathrm{d}\phi + \frac{\langle\pi\,\alpha\rangle^2\langle\hat{\pi}\,\alpha\rangle^2}{2\|\pi\|^4}[\mathrm{d}\phi,\phi] + \mathcal{O}\Big(\langle\pi\,\alpha\rangle^3\Big) \\ &= -\frac{\langle\pi\,\alpha\rangle}{\langle\pi\,\hat{\alpha}\rangle}\,\mathrm{d}\phi + \frac{\langle\pi\,\alpha\rangle^2}{2\langle\pi\,\hat{\alpha}\rangle^2}[\mathrm{d}\phi,\phi] + \mathcal{O}\Big(\langle\pi\,\alpha\rangle^3\Big)\,, \end{aligned} \tag{4.48}$$

where $\mathrm{d} = \mathrm{d}_{\mathbb{R}^4}$ denotes the exterior derivative on $\mathbb{R}^4$ and $\mathcal{O}\Big(\langle\pi\,\alpha\rangle^n\Big)$ indicates a term which vanishes to order n at $\pi \sim \alpha$. Consider

$$
\begin{aligned}
\hat{\sigma}\bar{\mathcal{A}}_A\hat{\sigma}^{-1} &= \pi^{A'}(\partial_{AA'}\hat{\sigma}\hat{\sigma}^{-1} + A_{AA'})\,,\\
&= -\frac{\langle\pi\,\alpha\rangle\langle\hat{\pi}\,\alpha\rangle}{\|\pi\|^2}\pi^{A'}\partial_{AA'}\phi + \pi^{A'}A_{AA'} + \mathcal{O}\Big(\langle\pi\,\alpha\rangle^2\Big)\,,\\
&= \langle\pi\,\hat{\alpha}\rangle\alpha^{A'}A_{AA'} + \langle\pi\,\alpha\rangle(\alpha^{A'}\partial_{AA'}\phi - \hat{\alpha}^{A'}A_{AA'}) + \mathcal{O}\Big(\langle\pi\,\alpha\rangle^2\Big)\,.
\end{aligned}
\tag{4.49}
$$

For $\bar{\mathcal{A}}$ to be divisible by $\langle\pi\,\alpha\rangle^2$ we require that

$$
\alpha^{A'}A_{AA'} = 0\,, \qquad \hat{\alpha}^{A'}A_{AA'} = \alpha^{A'}\partial_{AA'}\phi\,. \tag{4.50}
$$

The solution to the first is $A_{AA'} \sim \alpha_{A'}$, and then the second implies

$$
A_{AA'} = -\alpha_{A'}\alpha^{B'}\partial_{AB'}\phi\,. \tag{4.51}
$$

Here we have assumed without loss of generality that $\|\alpha\|^2 = \langle\alpha\,\hat{\alpha}\rangle = 1$. This can be viewed as a gauge fixing condition for an ASD gauge field, but was the case for the condition (4.38), it is not attainable for a generic connection. (See appendix B.5 for further details.)

We now show that HCS descends to the LMP action once we evaluate it on the above configuration. We first observe that

$$
\mathrm{HCS}(X+Y) = \mathrm{HCS}(X) + 2\mathrm{tr}(\bar{\mathcal{F}}(X)Y) - \bar{\partial}\mathrm{tr}(XY) + 2\mathrm{tr}(XY^2) + \mathrm{HCS}(Y)\,. \tag{4.52}
$$

Letting $\bar{\mathcal{J}} = -\bar{\partial}\hat{\sigma}\hat{\sigma}^{-1}$, and setting $X = \hat{\sigma}^{-1}\bar{\partial}\hat{\sigma} = -\hat{\sigma}^{-1}\bar{\mathcal{J}}\hat{\sigma}$ and $Y = \hat{\sigma}^{-1}\bar{\mathcal{A}}'\hat{\sigma}$ in the above gives

$$
\mathrm{HCS}(\bar{\mathcal{A}}) = \frac{1}{3}\mathrm{tr}(\bar{\mathcal{J}}^3) + \bar{\partial}\mathrm{tr}(\bar{\mathcal{J}}\bar{\mathcal{A}}') - 2\mathrm{tr}(\bar{\mathcal{J}}\bar{\mathcal{A}}'^2) + \mathrm{tr}(\hat{\sigma}^{-1}\bar{\mathcal{A}}'\hat{\sigma}\bar{\partial}(\hat{\sigma}^{-1}\bar{\mathcal{A}}'\hat{\sigma})). \tag{4.53}
$$

For the final term to contribute to the action, $\bar{\partial}$ must act as $\bar{e}^0\bar{\partial}_0$, since $\bar{\mathcal{A}}'$ has no $\bar{e}^0$ component. Furthermore, recalling that $\bar{\partial}_0\hat{\mathcal{A}}'_A = 0$, this $\bar{\partial}$ operator must in fact act on either $\hat{\sigma}$ or $\hat{\sigma}^{-1}$, so

$$
\mathrm{tr}(\hat{\sigma}^{-1}\bar{\mathcal{A}}'\hat{\sigma}\bar{\partial}(\hat{\sigma}^{-1}\bar{\mathcal{A}}'\hat{\sigma})) = -2\mathrm{tr}(\bar{\partial}\hat{\sigma}\hat{\sigma}^{-1}\bar{\mathcal{A}}'^2) = 2\mathrm{tr}(\bar{\mathcal{J}}\bar{\mathcal{A}}'^2)\,, \tag{4.54}
$$

which cancels the penultimate term. We conclude that

$$\mathrm{HCS}(\bar{\mathcal{A}}) = \frac{1}{3}\mathrm{tr}(\bar{\mathcal{J}}^3) + \bar{\partial}\mathrm{tr}(\bar{\mathcal{J}}\bar{\mathcal{A}}')\,. \tag{4.55}$$

Let's begin by considering the second of these terms. Its contribution to the action is

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{PT}}\Omega\wedge\bar{\partial}\mathrm{tr}(\bar{\mathcal{J}}\wedge\bar{\mathcal{A}}') &= \frac{1}{2\pi i}\int_{\mathbb{PT}}\bar{\partial}\Omega\wedge\mathrm{tr}(\bar{\mathcal{J}}\wedge\bar{\mathcal{A}}')\\ &= \frac{1}{4\pi i}\int_{\mathbb{CP}^1} e^0\wedge\bar{e}^0\,\bar{\partial}_0\bigg(\frac{1}{\langle\pi\,\alpha\rangle^4}\bigg)\int_{\mathbb{R}^4}\mathrm{d}^2x^{A'B'}\pi_{A'}\pi_{B'}\wedge\mathrm{tr}(\bar{\mathcal{J}}\wedge\bar{\mathcal{A}}')\,.\end{aligned} \tag{4.56}$$

We can interpret the integral over $\mathbb{CP}^1$ as extracting the coefficient of $\langle\pi\,\alpha\rangle^3$ in

$$\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{d}^2x^{A'B'}\pi_{A'}\pi_{B'}\wedge\mathrm{tr}(\bar{\mathcal{J}}\wedge\bar{\mathcal{A}}')\,. \tag{4.57}$$

To compute it we first note that

$$\mathrm{d}^2x^{A'B'}\pi_{A'}\pi_{B'}\wedge\bar{e}^A\wedge\bar{e}^B = -2\varepsilon^{AB}\mathrm{vol}_\delta\,, \tag{4.58}$$

allowing us to rewrite (4.57) as

$$\int_{\mathbb{R}^4}\mathrm{vol}_\delta\,\mathrm{tr}(\bar{\mathcal{J}}^A\bar{\mathcal{A}}'_A)\,. \tag{4.59}$$

We therefore need to expand $\mathrm{tr}(\bar{\mathcal{J}}^A\bar{\mathcal{A}}'_A)$ to order $\langle\pi\,\alpha\rangle^3$. From equation (4.51) we know that $\bar{\mathcal{A}}'_A$ vanishes to first order at $\pi\sim\alpha$, so it sufficient to expand $\bar{\mathcal{J}}$ to second order in $\langle\pi\,\alpha\rangle$. From (4.48)

$$\begin{aligned}\bar{\mathcal{J}}_A &= -\pi^{A'}\partial_{AA'}\hat{\sigma}\hat{\sigma}^{-1}\\ &= -\langle\pi\,\alpha\rangle\alpha^{A'}\partial_{AA'}\phi + \frac{\langle\pi\,\alpha\rangle^2}{\langle\pi\,\hat{\alpha}\rangle}\Big(\hat{\alpha}^{A'}\partial_{AA'}\phi + \frac{1}{2}[\alpha^{A'}\partial_{AA'}\phi,\phi]\Big) + \mathcal{O}\big(\langle\pi\,\alpha\rangle^3\big)\,.\end{aligned} \tag{4.60}$$

The coefficient of $\langle\pi\,\alpha\rangle^3$ in $\mathrm{tr}(\bar{\mathcal{J}}^A\bar{\mathcal{A}}'_A)$ is therefore

$$\varepsilon^{AB}\mathrm{tr}\bigg(\Big(\hat{\alpha}^{A'}\partial_{AA'}\phi + \frac{1}{2}[\alpha^{A'}\partial_{AA'}\phi,\phi]\Big)\alpha^{B'}\partial_{BB'}\phi\bigg)\,. \tag{4.61}$$

We conclude that contribution of (4.57) to the effective spacetime action is

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{vol}_\delta\,\varepsilon^{AB}\Big(\varepsilon^{A'B'}\mathrm{tr}(\partial_{AA'}\phi\partial_{BB'}\phi)-\alpha^{A'}\alpha^{B'}\mathrm{tr}(\phi[\partial_{AA'}\phi,\partial_{BB'}\phi])\Big)\\ =\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{tr}\Big(\,\mathrm{d}\phi\wedge*\,\mathrm{d}\phi+\omega_{\alpha,\alpha}\wedge\phi\,\mathrm{d}\phi\wedge\mathrm{d}\phi\Big)\,,\end{aligned}\tag{4.62}$$

where $\omega_{\alpha,\alpha}=\mathrm{d}^2x^{A'B'}\alpha_{A'}\alpha_{B'}$.

Now let us turn out attention to the first term in equation (4.55), given by

$$\frac{1}{12\pi i}\int_{\mathbb{PT}}\Omega\wedge\mathrm{tr}(\bar{\mathcal{J}}^3)=\frac{1}{12\pi i}\int_{\mathbb{CP}^1}\frac{e^0\wedge\bar{e}^0}{\langle\pi\,\alpha\rangle^4}\int_{\mathbb{R}^4}\mathrm{d}^2x^{A'B'}\pi_{A'}\pi_{B'}\,\mathrm{tr}(\bar{\mathcal{J}}^3)\,.\tag{4.63}$$

Consider the U(1) action on $\mathbb{CP}^1$ preserving $\alpha,\hat{\alpha}$, acting as $\langle\pi\,\alpha\rangle\mapsto e^{i\theta}\langle\pi\,\alpha\rangle$, $\langle\pi\,\hat{\alpha}\rangle\mapsto\langle\pi\,\hat{\alpha}\rangle$. Under this action, the argument of the exponential in the definition of $\hat{\sigma}$ has positive charge (4.46). We deduce that expanding out $\bar{\mathcal{J}}$ using Dunhamel's formula generates terms which also have positive charge. The integral over the orbits of the U(1) action will pick out the invariant part of the integrand. It is therefore clear that it only receives contributions from the coefficient of $\langle\pi\,\alpha\rangle^4$ in

$$\int_{\mathbb{R}^4}\mathrm{d}^2x^{A'B'}\pi_{A'}\pi_{B'}\wedge\mathrm{tr}(\bar{\mathcal{J}}^3)\,.\tag{4.64}$$

Now from (4.46) $\bar{\mathcal{J}}_0=\langle\pi\,\alpha\rangle^2\phi+\mathcal{O}\Big(\langle\pi\,\alpha\rangle^3\Big)$, and from (4.60) $\bar{\mathcal{J}}_A=-\langle\pi\,\alpha\rangle\alpha^{A'}\partial_{AA'}\phi+\mathcal{O}\Big(\langle\pi\,\alpha\rangle^2\Big)$. We therefore conclude that (4.63) contributes

$$\begin{aligned}&-\frac{1}{2\pi i}\int_{\mathbb{CP}^1}e^0\wedge\bar{e}^0\int_{\mathbb{R}^4}\mathrm{vol}_\delta\,\varepsilon^{AB}\alpha^{A'}\alpha^{B'}\mathrm{tr}(\phi\partial_{AA'}\phi\partial_{BB'}\phi)\\ &\quad=\frac{1}{6}\int_{\mathbb{R}^4}\mathrm{vol}_\delta\,\varepsilon^{AB}\alpha^{A'}\alpha^{B'}\mathrm{tr}(\phi[\partial_{AA'}\phi,\partial_{BB'}\phi])=-\frac{1}{6}\int_{\mathbb{R}^4}\omega_{\alpha,\alpha}\wedge\mathrm{tr}(\phi\,\mathrm{d}\phi\wedge\mathrm{d}\phi)\,.\end{aligned}\tag{4.65}$$

Combining equations (4.62) & (4.65) gives the LMP action

$$S_{\mathrm{LMP}}[\phi]=\int_{\mathbb{R}^4}\mathrm{tr}\left(\frac{1}{2}\,\mathrm{d}\phi\wedge*\,\mathrm{d}\phi+\frac{1}{3}\omega_{\alpha,\alpha}\wedge\phi\,\mathrm{d}\phi\wedge\mathrm{d}\phi\right),\tag{4.66}$$

originally appearing in [105, 104, 139]. Its classical equation of motion is

$$\mathrm{d}*\mathrm{d}\phi=\omega_{\alpha,\alpha}\wedge\mathrm{d}\phi\wedge\mathrm{d}\phi\,,\tag{4.67}$$

which is equivalent to the ASDYM equations for $A_{AA'}=-\alpha_{A'}\alpha^{B'}\partial_{AB'}\phi$ (see appendix B.5).

We make the following observations:

- Choosing Ω to have a fourth order pole clearly requires the fewest arbitrary choices of dual twistors, and so breaks conformal invariance in the least damaging way. Indeed, the Lagrangian of HCS is invariant under the subgroup of the complexified conformal group $\mathrm{PSL}_4(\mathbb{C})$ preserving the dual twistor A. This includes all translations and right handed rotations, but also the combined dilations and left handed rotations acting as

$$\delta x^{AA'} = \frac{1}{2}(\alpha^{A'}\beta_{B'} - 3\beta^{A'}\alpha_{B'})x^{AB'}\,, \qquad \delta\pi_{A'} = \alpha_{A'}\beta^{B'}\pi_{B'}\,, \tag{4.68}$$

  and the special conformal transformations acting as

$$\delta x^{AA'} = \frac{1}{2}x^2\lambda^A\alpha^{A'} - x^{AA'}x^{BB'}\lambda_B\alpha_{B'}\,, \qquad \delta\pi_{A'} = \alpha_{A'}x^{BB'}\lambda_B\pi_{B'}\,, \tag{4.69}$$

  for left and right handed spinors β and λ respectively. Unfortunately in Euclidean signature the reality conditions on x mean that only the translations and right handed rotations are realised as symmetries of the LMP action. We shall see in section 4.5 that LMP action only admits natural reality conditions on ϕ in ultrahyperbolic signature, for which subgroups of all of the above spacetime symmetries survive.

- In [34] it was observed that a supersymmetric (SUSY) analogue of the LMP action could be used to describe ASD $\mathcal{N} = 4$ super Yang-Mills. This arises from a twistor action

$$\frac{1}{2\pi i}\int_{\mathbb{CP}^3\times\mathbb{C}^{0|4}} \frac{\mathrm{D}^3Z\wedge \mathrm{d}^4\chi}{\langle\pi\,\alpha\rangle^4}\wedge \mathrm{HCS}(\bar{\mathcal{A}}) \tag{4.70}$$

  where $\bar{\mathcal{A}}$ is required to be divisible by $\langle\pi\,\alpha\rangle^2$ as above. Here $\mathbb{C}^{0|4}$ refers to the complex 4 dimensional odd super vector space with coordinates χ^m for $m = 1,\dots,4$. The SUSY LMP action was demonstrated in [34] to be equivalent to the standard Lorentz invariant action for ASD $\mathcal{N} = 4$ super Yang-Mills on spacetime. At the level of the Lagrangian this equivalence is clear from the twistor description, since we can rewrite the above as an integral over $\mathbb{CP}^{3|4}$ by mapping

$$\chi^a \mapsto \psi^a = \langle\pi\,\alpha\rangle\chi^a\,. \tag{4.71}$$

  This recovers the familiar twistor action of Witten [169, 32]. (A similar supersymmetric analogue of WZW_4 was also identified in [34], which can presumably also be written on twistor space using a minor modification of the construction described in subsection 4.3.1.)

4.3.3 Trigonometric action

We now derive an alternative action for the ASDYM equations which we refer to as the trigonometric action, since the boundary conditions we impose are the same as those which lead to trigonometric R-matrices in CS_4. We will show that it is closely related to an action Mason and Sparling [124].

We take

$$\Omega = \frac{\mathrm{D}^3 Z}{(Z\cdot A_+)(Z\cdot A_-)(Z\cdot B_+)(Z\cdot B_-)} = \frac{e^0\wedge e^A\wedge e_A}{2\langle\pi\,\alpha_+\rangle\langle\pi\,\alpha_-\rangle\langle\pi\,\beta_+\rangle\langle\pi\,\beta_-\rangle}\,, \tag{4.72}$$

and assume without loss of generality that $\langle\alpha_+\,\beta_+\rangle = \langle\alpha_-\,\beta_-\rangle = 1$. To be able to choose the dual twistors $A_\pm, B_\pm$ so that they are simultaneously represented by $(0,\alpha_\pm), (0,\beta_\pm)$ for left handed spinors $\alpha_\pm, \beta_\pm$ the loci $A_\pm\cdot Z = 0$ and $B_\pm\cdot Z = 0$ must intersect on the single $\mathbb{CP}^1$ excised from $\mathbb{CP}^3$.

Varying the action on the support of the bulk equations of motion generates boundary terms at the locations of the poles. For example, the unwanted boundary term at $\pi\sim\alpha_+$ is

$$\frac{1}{2\langle\alpha_+\,\alpha_-\rangle\langle\alpha_+\,\beta_-\rangle}\int_{\mathbb{R}^4}\omega_{\alpha_+,\alpha_+}\wedge\mathrm{tr}(\delta\bar{\mathcal{A}}\wedge\bar{\mathcal{A}})|_{\pi\sim\alpha_+}\,. \tag{4.73}$$

As was observed for CS_4 in subsection 1.3.5, requiring that $\bar{\mathcal{A}}$ be divisible by $\langle\pi\,\alpha_+\rangle$ would introduce a double zero in $\mathrm{tr}(\delta\bar{\mathcal{A}}\wedge\bar{\mathcal{A}})$ at $\pi\sim\alpha_+$, a needlessly stringent condition. In this section we instead impose the trigonometric boundary conditions, originally proposed in [50] and first applied in the context of 2d CIFT in [53]. We reviewed these in the context of CS_4 in section 1.3.5, but briefly recap them here for convenience. This discussion largely follows that appearing in [53] for the analogous boundary conditions in CS_4.

We assume that our Lie algebra $\mathfrak{g}$ is a Manin triple, i.e. it splits as

$$\mathfrak{g}\cong\mathfrak{g}_-\dotplus\mathfrak{g}_+ \tag{4.74}$$

for $\mathfrak{g}_\pm$ disjoint Lagrangian subalgebras of $\mathfrak{g}$. To construct examples of Manin triples we choose a complex simple Lie algebra $\mathfrak{g}_0$ which comes with a minimal $\mathfrak{g}_0$-invariant bilinear tr_0. Fixing a choice of Cartan subalgebra and base we have the standard decomposition $\mathfrak{g}_0 = \mathfrak{n}_- \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+$. We then define

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \widetilde{\mathfrak{h}}\,, \tag{4.75}$$

where $\widetilde{\mathfrak{h}}$ is a second copy of the Cartan, and extend tr_0 to $\mathfrak{g}$ by

$$\mathrm{tr}_{\mathfrak{g}} = \mathrm{tr}_{\mathfrak{g}_0} - \mathrm{tr}_{\mathfrak{g}_0}|_{\widetilde{\mathfrak{h}}}\,. \tag{4.76}$$

The Lagrangian subalgebras of the Manin triple are

$$\mathfrak{g}_- = \mathfrak{n}_- \dotplus \mathfrak{h}_-\,, \qquad \mathfrak{g}_+ = \mathfrak{n}_+ \dotplus \mathfrak{h}_+ \tag{4.77}$$

for $\mathfrak{h}_+$ and $\mathfrak{h}_-$ the diagonal and antidiagonal subalgebras in $\mathfrak{h} \oplus \widetilde{\mathfrak{h}}$. Note that our choice of bilinear and Lagrangian subalgebras differ slightly from those used in subsection 1.3.5.

We can also perform this decomposition at the level of the Lie group. In an open subset, U of G_0 (the Lie group with Lie algebra $\mathfrak{g}_0$) we have the decomposition $G_0 = N_- H N_+$. U is known as the big Bruhat cell, and within it this decomposition is unique. We then have $G = G_0 \times \widetilde{H}$, and we can identify the subgroups $G_\pm$ with Lie algebras $\mathfrak{g}_\pm$. They are

$$G_- = N_- H_-\,, \qquad G_+ = H_+ N_+\,, \tag{4.78}$$

where $H_\pm$ are subgroups of G with Lie algebras $\mathfrak{h}_\pm$.

The natural map

$$G_- \times G_+ \to U \times \widetilde{H} \subset G \tag{4.79}$$

is a $2^{\mathrm{rank}\,\mathfrak{g}_0}$-fold cover. This is because in inverting it we must take the square root of an element of $\mathfrak{h}$.

Given a gauge group G of this form, we enforce the following boundary conditions on $\bar{\mathcal{A}}$ at $\alpha_\pm, \beta_\pm$

$$\bar{\mathcal{A}}_A|_{\pi\sim\alpha_\pm}, \bar{\mathcal{A}}_A|_{\pi\sim\beta_\pm} \in \mathfrak{g}_\mp\,, \tag{4.80}$$

where the unusual choice of signs is for later convenience. These eliminate the boundary terms by virtue of the fact that $\mathfrak{g}_\pm$ are Lagrangian. For consistency with gauge transformations we also require that

$$\varepsilon|_{\pi\sim\alpha_\pm}, \varepsilon|_{\pi\sim\beta_\pm} \in \mathfrak{g}_\mp\,. \tag{4.81}$$

We now determine the effective spacetime theory. As usual, we begin by introducing $\hat{\sigma} : \mathbb{PT} \to G$. Since gauge transformations are arbitrary away from $\pi \sim \alpha_\pm, \beta_\pm$, $\hat{\sigma}$ is determined by its values $\sigma_{\alpha_\pm} = \hat{\sigma}|_{\pi\sim\alpha_\pm}$ and $\sigma_{\beta_\pm} = \hat{\sigma}|_{\pi\sim\beta_\pm}$ at these points. Under a gauge transformation $\sigma_{\alpha_\pm} \mapsto \sigma_{\alpha_\pm} g^{-1}|_{\pi\sim\alpha_\pm}$ where $g|_{\pi\sim\alpha_\pm}$ take values in $G_\mp$. Assuming that the image of σ_{α_+} is contained in $U \times \widetilde{H}$, we can partially fix the gauge by demanding that σ_{α_+} maps into G_+. Making the same assumptions about the images of σ_{α_-} and $\sigma_{\beta_\pm}$, we can completely fix the gauge by requiring that $\sigma_{\alpha_\pm}, \sigma_{\beta_\pm}$ map into $G_\pm$ respectively.

This leaves the redundancy $\hat{\sigma} \mapsto h\hat{\sigma}$. To fix this we take $h = h_- h_+$ for $h_\pm$ taking values in $G_\pm$. We then choose h_+ so that $\sigma_{\beta_+} = \mathrm{id.}$ and similarly choose h_- so that $\sigma_{\beta_-} = \mathrm{id.}$. (When performing these shifts we must simultaneously gauge transform to ensure that the gauge fixing conditions on $\sigma_{\alpha_\pm}$ continue to hold.) This completely fixes the redundancy in $\hat{\sigma}$, and we are left with two spacetime fields $\sigma_\pm = \sigma_{\alpha_\pm}$ taking values in $G_\pm$ respectively.

Next, we turn our attention to the remaining components of $\bar{\mathcal{A}}$. Since Ω is nowhere vanishing we have

$$\bar{\mathcal{A}}'_A = \pi^{A'} A_{AA'} \tag{4.82}$$

and our boundary conditions then fix

$$A_{AA'} = -\beta_{+A'} \alpha_+{}^{B'} \partial_{AB'} \sigma_+ \sigma_+^{-1} - \beta_{-A'} \alpha_-{}^{B'} \partial_{AB'} \sigma_- \sigma_-^{-1} \,. \tag{4.83}$$

Note that a general ASD connection can always be written in this form: one first projects onto $\mathfrak{g}_\pm$, and then introduces a Yang matrix $\sigma_\pm$ with values in $G_\pm$ for this projection.

Substituting this expression for $\bar{\mathcal{A}}'$ back into the action we obtain an effective action in terms of $\sigma_\pm$. This computation is essentially identical to that appearing in subsection 4.3.2 and similar ones for CS_4 in [53], so we omit the details here. It gives

$$S[\sigma_\pm] = \frac{1}{\langle \alpha_+ \, \alpha_- \rangle} \int_{\mathbb{R}^4} \mathrm{vol}_\delta \, \varepsilon^{AB} \alpha_-^{A'} \alpha_+^{B'} \mathrm{tr}(J_{-AA'} J_{+BB'}) \,, \tag{4.84}$$

where $J_- = -\,\mathrm{d}\sigma_- \sigma_-^{-1}$ and $J_+ = -\,\mathrm{d}\sigma_+ \sigma_+^{-1}$. Introducing the projections

$$\pi_a{}^b = \langle \alpha_- \, \alpha_+ \rangle^{-1} \delta_A{}^B \alpha_{+A'} \alpha_-^{B'} \qquad \tilde{\pi}_a{}^b = -\langle \alpha_- \, \alpha_+ \rangle^{-1} \delta_A{}^B \alpha_{-A'} \alpha_+^{B'} \tag{4.85}$$

and associated operators $\partial = \mathrm{d}x^a \pi_a{}^b \partial_b$, $\tilde{\partial} = \mathrm{d}x^a \tilde{\pi}_a{}^b \partial_b$ we can rewrite the action as

$$S[\sigma_\pm] = \frac{1}{\langle \alpha_- \, \alpha_+ \rangle} \int_{\mathbb{R}^4} \omega_{\alpha_-,\alpha_+} \wedge \mathrm{tr}(\partial \sigma_- \sigma_-^{-1} \, \tilde{\partial} \sigma_+ \sigma_+^{-1}) \,. \tag{4.86}$$

If $\alpha_+ = \hat{\alpha}_-$, then $\partial, \tilde{\partial} = \bar{\partial}$ coincide with the Dolbeault operators, and $\omega_{\alpha_-,\hat{\alpha}_-}$ with the Kähler form (up to an overall factor), in the complex structure on spacetime determined by α_-.

The corresponding classical equations of motion are

$$\Big[\omega_{\alpha_-,\alpha_+} \wedge \partial(\sigma_-^{-1} \tilde{\partial} \sigma_+ \sigma_+^{-1} \sigma_-)\Big]_{\mathfrak{g}_+} = 0 \,, \quad \Big[\omega_{\alpha_-,\alpha_+} \wedge \tilde{\partial}(\sigma_+^{-1} \partial \sigma_- \sigma_-^{-1} \sigma_+)\Big]_{\mathfrak{g}_-} = 0 \,, \tag{4.87}$$

where the square brackets indicate that we are projecting onto the indicated subalgebra. These equations are in fact equivalent to the ASDYM equations for a $\mathfrak{g}$-valued gauge field, a proof of which is included in appendix A.6.

We can also understand these equations of motion in terms of fields taking values in the simple Lie group G_0. To do so we write

$$\sigma_-^{-1} = (\ell h_-, h_-^{-1}) \,, \quad \sigma_+ = (h_+ u, h_+) \,, \tag{4.88}$$

where $h_\pm$ take values in H, and u, ℓ take values in $N_\pm$ respectively. Projecting Yang's equation onto $\tilde{\mathfrak{h}}$, which is abelian, we find that

$$\omega_{\alpha_-,\alpha_+} \wedge \partial(\tilde{\partial}(h_-^{-1} h_+) h_+^{-1} h_-) = \omega_{\alpha_-,\alpha_+} \wedge \partial(\tilde{\partial} h_+ h_+^{-1} - \tilde{\partial} h_- h_-^{-1}) = 0 \,. \tag{4.89}$$

Under the assumption that $h_\pm \to \mathrm{id.}$ as $x^2 \to \infty$ in $\mathbb{R}^4$, the solution to this equation is $h_+ = h_- = h$. We are left with Yang's equation for $\ell h^2 u$.

Imposing $h_\pm = h$ at the level of the action gives

$$S[L, U] = \frac{1}{\langle \alpha_- \, \alpha_+ \rangle} \int_{\mathbb{R}^4} \omega_{\alpha_-,\alpha_+} \wedge \mathrm{tr}_0 \bigg(\partial L L^{-1} \wedge \tilde{\partial} U U^{-1} - \frac{1}{2} \partial L L^{-1} \wedge \tilde{\partial} L L^{-1} \bigg) \tag{4.90}$$

where $u = U \in N_+$ and $L = h^{-2} \ell^{-1} \in B_- = N_- H$. A derivation is included in appendix A.6. This is an action for the ASDYM equations due to Mason and Sparling [124].

We note the following:

- Off-shell, the action (4.84) agrees with that of WZW$_4$ (4.39) for the group G. To see this, decompose the dynamical field of WZW$_4$ as $\sigma = \sigma_-^{-1} \sigma_+$ for $\sigma_\pm \in G_\pm$, and then apply the Polyakov-Wiegmann identity. Only the cross term involving both of $\sigma_\pm$ survives, the remaining terms vanishing on account of the fact that $\mathfrak{g}_\pm$ are Lagrangian. Curiously this agreement is not manifest on twistor space. The same observation can be made in CS_4.

- In [60] generalisations of the trigonometric boundary conditions appearing in [53] were introduced. (We briefly discussed these at the end of subsection 1.3.5.) These lead to, among other things, the λ- and η-deformations. Analogues of these boundary conditions applied to HCS theory have since been studied in [36], largely with the motivation of recovering their description in CS_4 from a symmetry reduction. In the process, a number of 4d analogues of these deformations were obtained, albeit often with restricted parameters.

4.4 4d integrable coupled σ-models

So far we have only considered Ω which are nowhere vanishing. We now briefly consider what happens if we relax this constraint. The effective spacetime actions we obtain do not have classical equations of motion equivalent to the ASDYM equations. They do, however, admit a 4d Lax connection, and we will later see their symmetry reductions describe known 2d integrable theories.

We restrict our attention to the following, fairly general, choice of Ω

$$\Omega = \frac{\mathrm{D}^3 Z \prod_{j=1}^n (A_j \cdot Z)(B_j \cdot Z)}{\prod_{i=1}^{n+2} (C_i \cdot Z)^2} = \frac{e^0 \wedge e^A \wedge e_A \prod_{j=1}^n \langle \pi\, \alpha_j \rangle \langle \pi\, \beta_j \rangle}{2 \prod_{i=1}^{n+2} \langle \pi\, \gamma_i \rangle^2} \,. \tag{4.91}$$

For the second equality to hold the loci $A_j \cdot Z = 0$, $B_j \cdot Z = 0$ and $C_i \cdot Z = 0$ must all intersect on the $\mathbb{CP}^1$ at infinity. We introduce the standard boundary conditions for $\bar{\mathcal{A}}$ at the double poles of Ω, i.e. that $\bar{\mathcal{A}}$ is divisible by $\prod_{i=1}^{n+2} \langle \pi\, \gamma_i \rangle$. At zeros of Ω we can tolerate simple poles in components of $\bar{\mathcal{A}}_A$ without generating boundary terms when varying the action. More precisely, introducing the right-handed dyad $[\mu\, \nu] = 1$, we permit simple poles in $\mu^A \bar{\mathcal{A}}_A$ and $\nu^A \bar{\mathcal{A}}_A$ at $\pi \sim \alpha_j$ and $\pi \sim \beta_j$ respectively for all $j = 1, \ldots, n$. Note that the combination $\Omega \wedge \mathrm{tr}(\delta \bar{\mathcal{A}} \wedge \bar{\mathcal{A}})$, which appears when varying the action on the support of the equations of motion, is free from poles.

The pullback of $\bar{\mathcal{A}}$ to the twistor fibres $\iota_x^* \bar{\mathcal{A}}$ has no poles, and so we can introduce $\hat{\sigma} : \mathbb{PT} \to G$ in the usual way. The gauge invariant data that can be extracted from $\iota_x^* \bar{\mathcal{A}}$ is furnished by the holomorphic Wilson lines between the γ_i, and can be completely characterised by the map

$$\sigma : \mathbb{R}^4 \to G \backslash G^{n+2} \,, \qquad x \mapsto [(\sigma_1, \ldots, \sigma_{n+2})] = [(h\sigma_1, \ldots, h\sigma_{n+2})] \tag{4.92}$$

with $\sigma_i = \hat{\sigma}|_{\pi\sim\gamma_i}$. We could identify $G\backslash G^{n+2}$ with G^{n+1} by fixing $\sigma_{n+2} = \mathrm{id.}$, but we will find it convenient not to do so. We assume $\hat{\sigma}$ takes the value σ_i in a neighbourhood of γ_i for all $i = 1, \ldots, n+2$. The gauge invariant holomorphic Wilson lines are

$$\mathcal{W}_{\gamma_i \to \gamma_j} = \hat{\sigma}^{-1}|_{\pi\sim\gamma_j}\hat{\sigma}|_{\pi\sim\gamma_i} = \sigma_j^{-1}\sigma_i \,. \tag{4.93}$$

The redundancy $\sigma_i \mapsto h\sigma_i$ will be a gauge symmetry of the resulting 4d theory on spacetime.

The next step is to solve the classical equations of motion in the directions of the fibres. As usual we write

$$\bar{\mathcal{A}} = \hat{\sigma}^{-1}\bar{\partial}\hat{\sigma} + \hat{\sigma}^{-1}\bar{\mathcal{A}}'\hat{\sigma} \,, \tag{4.94}$$

but where $\bar{\mathcal{A}}'_A$ is now meromorphic in π. Simple poles are permitted at the α_j in $\mu^A\bar{\mathcal{A}}_A$ and at the β_j in $\nu^A\bar{\mathcal{A}}_A$. Our boundary conditions imply $\bar{\mathcal{A}}|_{\pi\sim\gamma_i} = 0$ for $i = 1\ldots n+2$, hence

$$\bar{\mathcal{A}}'_A|_{\pi\sim\gamma_i} = -\left.\frac{\langle\pi\,\hat{\gamma}_i\rangle\gamma_i^{A'}\partial_{AA'}\sigma_i\sigma_i^{-1}}{\|\gamma_i\|^2}\right|_{\pi\sim\gamma_i} . \tag{4.95}$$

The unique choice for $\bar{\mathcal{A}}'$ obeying these constraints is

$$\bar{\mathcal{A}}'_A = \sum_{i=1}^{n+2}\prod_{j=1,\,j\neq i}^{n+2}\frac{\langle\pi\,\gamma_j\rangle}{\langle\gamma_i\,\gamma_j\rangle}\left(\nu_A\mu^B\prod_{k=1}^{n}\frac{\langle\gamma_i\,\alpha_k\rangle}{\langle\pi\,\alpha_k\rangle} - \mu_A\nu^B\prod_{k=1}^{n}\frac{\langle\gamma_i\,\beta_k\rangle}{\langle\pi\,\beta_k\rangle}\right)\gamma_i^{B'}J_{iBB'} \,, \tag{4.96}$$

where we have defined $J_i = -\,\mathrm{d}\sigma_i\sigma_i^{-1}$. Note that it is essential that we allow simple poles in $\bar{\mathcal{A}}$ for us to be able to satisfy the boundary conditions. We also emphasise that $\bar{\mathcal{A}}'$ is not linear in $\pi_{A'}$ for $n > 0$, and so cannot be straightforwardly related to a spacetime gauge field. We can, however, still interpret $\bar{\mathcal{A}}'_A = \mathscr{L}_A$ as a 4d Lax connection.

Under the redundancy $(\sigma_1, \ldots, \sigma_{n+2}) \mapsto (h\sigma_1, \ldots, h\sigma_{n+2})$ we have

$$\mathscr{L}_A \mapsto h\mathscr{L}_Ah^{-1} - \sum_{i=1}^{n+2}\prod_{j=1,\,j\neq i}^{n+2}\frac{\langle\pi\,\gamma_j\rangle}{\langle\gamma_i\,\gamma_j\rangle}\left(\nu_A\mu^B\prod_{k=1}^{n}\frac{\langle\gamma_i\,\alpha_k\rangle}{\langle\pi\,\alpha_k\rangle} - \mu_A\nu^B\prod_{k=1}^{n}\frac{\langle\gamma_i\,\beta_k\rangle}{\langle\pi\,\beta_k\rangle}\right)\gamma_i^{B'}\partial_{BB'}hh^{-1} \,. \tag{4.97}$$

To simplify this we use

$$\sum_{i=1}^{n+2}\prod_{j=1,\,j\neq i}^{n+2}\frac{\langle\pi\,\gamma_j\rangle}{\langle\gamma_i\,\gamma_j\rangle}\prod_{k=1}^{n}\frac{\langle\gamma_i\,\alpha_k\rangle}{\langle\pi\,\alpha_k\rangle}\gamma_i^{A'} = \pi^{A'} \,, \tag{4.98}$$

which can be verified by evaluating both sides at $\pi \sim \alpha_i$, and computing the residues at $\pi \sim \alpha_k$. The same identity holds if we replace α by β. We therefore deduce that

$$\mathscr{L}_A \mapsto h\mathscr{L}_A h^{-1} - \pi^{A'}\partial_{AA'}hh^{-1} \tag{4.99}$$

as expected.

Following by now fairly standard methods we can determine the effective spacetime action. It is given by

$$\begin{aligned}&\sum_{i=1}^{n+2} a_i b_i \Bigg(\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}(J_i \wedge * J_i) - \sum_{k=1}^{n} \frac{\langle \alpha_k\, \beta_k\rangle}{\langle \alpha_k\, \gamma_i\rangle\langle \gamma_i\, \beta_k\rangle} \int_{\mathbb{R}^4} \mathrm{vol}_\delta\, \mu^{(A}\nu^{B)}\gamma_i^{A'}\gamma_i^{B'}\mathrm{tr}(J_{iAA'}J_{iBB'}) \\ &\quad + \frac{1}{3}\int_{\mathbb{R}^4\times[0,1]} \mathrm{d}^2x^{A'B'}\gamma_{iA'}\Bigg(\sum_{j=1\, j\neq i}^{n+2} \frac{\gamma_{jB'}}{\langle \gamma_i\, \gamma_j\rangle} - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\alpha_{kB'}}{\langle \gamma_i\, \alpha_k\rangle} + \frac{\beta_{kB'}}{\langle \gamma_i\, \beta_k\rangle}\right)\Bigg) \wedge \mathrm{tr}(\tilde{J}_i^3)\Bigg) \\ &\quad + \sum_{i,j=1\, i\neq j}^{n+2} \frac{1}{\langle \gamma_i\, \gamma_j\rangle}\int_{\mathbb{R}^4} \mathrm{vol}_\delta \left(a_i b_j \mu^A\nu^B - a_j b_i \mu^B \nu^A\right)\gamma_i^{A'}\gamma_j^{B'}\mathrm{tr}(J_{iAA'}J_{jBB'})\,. \end{aligned} \tag{4.100}$$

Here we have introduced

$$a_i = \frac{\prod_{k=1}^{n}\langle \gamma_i\, \alpha_k\rangle}{\prod_{j=1,\, j\neq i}^{n+2}\langle \gamma_i\, \gamma_j\rangle}\,, \quad b_i = \frac{\prod_{k=1}^{n}\langle \gamma_i\, \beta_k\rangle}{\prod_{j=1,\, j\neq i}^{n+2}\langle \gamma_i\, \gamma_j\rangle}\,. \tag{4.101}$$

$\tilde{\sigma}$ is a smooth homotopy from σ to the diagonal in $G\backslash G^{n+2}$, and as usual $\tilde{J}_i = -\tilde{\mathrm{d}}\tilde{\sigma}_i\tilde{\sigma}_i^{-1}$ for $\tilde{\mathrm{d}}$ the exterior derivative on $\mathbb{R}^4 \times [0,1]$. There are a couple of sanity checks we can perform to confirm whether this is the correct action.

- The first is that it's invariant under the redundancy $(\sigma_1, \ldots, \sigma_{n+2}) \mapsto (h\sigma_1, \ldots, h\sigma_{n+2})$. Infinitesimally we have

$$\delta\sigma_i = \varepsilon\sigma_i \tag{4.102}$$

  and so

$$\delta J_i = -\delta(\mathrm{d}\sigma_i\sigma_i^{-1}) = -\,\mathrm{d}\varepsilon - [\varepsilon, \mathrm{d}\sigma_i\sigma_i^{-1}] = -\,\mathrm{d}\varepsilon - [\varepsilon, J_i]\,. \tag{4.103}$$

  It is clear that the action is invariant under the simultaneous adjoint action on the currents J_i, and so it is enough to establish invariance under

$$\delta J_i = -\,\mathrm{d}\epsilon\,. \tag{4.104}$$

We find that[4]

$$\delta S = 2\sum_{i=1}^{n} a_i b_i \int_{\mathbb{R}^4} \mathrm{vol}_\delta\, \mathrm{tr}(\partial_{AA'}\varepsilon J_{iBB'})\gamma_i^{B'}\Bigg(\mu^A\nu^B\Bigg(\sum_{j=1,\, j\neq i}^{n+2}\frac{\gamma_j^{A'}}{\langle\gamma_i\,\gamma_j\rangle}\bigg(1+\frac{a_j}{a_i}\bigg) - \sum_{k=1}^{n}\frac{\alpha_k^{A'}}{\langle\gamma_i\,\alpha_k\rangle}\Bigg) - \mu^B\nu^A\Bigg(\sum_{j=1,\, j\neq i}^{n+2}\frac{\gamma_j^{A'}}{\langle\gamma_i\,\gamma_j\rangle}\bigg(1+\frac{b_j}{b_i}\bigg) - \sum_{k=1}^{n}\frac{\beta_k^{A'}}{\langle\gamma_i\,\beta_k\rangle}\Bigg)\Bigg)\,, \tag{4.107}$$

which vanishes since

$$\sum_{j=1,\, j\neq i}^{n+2}\frac{\gamma_j^{A'}}{\langle\gamma_i\,\gamma_j\rangle}\bigg(1+\frac{a_j}{a_i}\bigg) = \sum_{k=1}^{n}\frac{\alpha_k^{A'}}{\langle\gamma_i\,\alpha_k\rangle}\,, \tag{4.108}$$

and similarly if we replace all αs with βs.

- The second is that taking the limit $\alpha_n, \beta_n \to \gamma_{n+2}$ effectively reduces n to $n-1$. This property is manifest for the twistor action. On spacetime it follows from the fact that in this limit $a_{n+2} = b_{n+2} = 0$, and

$$\frac{\gamma_{n+2}^{A'}}{\langle\gamma_i\,\gamma_{n+2}\rangle} = \frac{\alpha_n^{A'}}{\langle\gamma_i\,\alpha_n\rangle} = \frac{\beta_n^{A'}}{\langle\gamma_i\,\beta_n\rangle}\,. \tag{4.109}$$

The classical equations of motion are

$$\sum_{j=1,\, j\neq i}^{n+2}\frac{\gamma_i^{A'}\gamma_j^{B'}}{\langle\gamma_i\,\gamma_j\rangle}(a_i b_j\mu^A\nu^B - a_j b_i\mu^B\nu^A)\Big(\partial_{AA'}J_{jBB'} - \partial_{BB'}J_{iAA'} + [J_{iAA'}, J_{jBB'}]\Big) = 0 \tag{4.110}$$

for $i = 1, \dots, n+2$. Note that the gauge symmetry is manifest.

We can understand these equations of motion by working directly with the Lax equation

$$\bar{\partial}_A\mathscr{L}^A + \frac{1}{2}[\mathscr{L}_A, \mathscr{L}^A] = 0\,. \tag{4.111}$$

[4]In deriving this we have made use of the spinor identities

$$\frac{\hat{\gamma}_i^{A'}}{\|\gamma_i\|^2} - \Bigg(\sum_{j=1,\, j\neq i}^{n+2}\frac{\langle\gamma_j\,\hat{\gamma}_i\rangle}{\langle\gamma_j\,\gamma_i\rangle} - \sum_{k=1}^{n}\frac{\langle\alpha_k\,\hat{\gamma}_i\rangle}{\langle\alpha_k\,\gamma_i\rangle}\Bigg)\frac{\gamma_i^{A'}}{\|\gamma_i\|^2} = \sum_{j=1,\, j\neq i}^{n+2}\frac{\gamma_j^{A'}}{\langle\gamma_i\,\gamma_j\rangle} - \sum_{k=1}^{n}\frac{\alpha_k^{A'}}{\langle\gamma_i\,\alpha_k\rangle}\,, \tag{4.105}$$

the same identity with α_j replaced β_j, and

$$\sum_{k=1}^{n}\frac{\langle\alpha_k\,\beta_k\rangle}{\langle\alpha_k\,\gamma_i\rangle\langle\gamma_i\,\beta_k\rangle} = \frac{1}{\|\gamma_i\|^2}\Bigg(\sum_{k=1}^{n}\frac{\langle\alpha_k\,\hat{\gamma}_i\rangle}{\langle\alpha_k\,\gamma_i\rangle} - \sum_{k=1}^{n}\frac{\langle\beta_k\,\hat{\gamma}_i\rangle}{\langle\beta_k\,\gamma_i\rangle}\Bigg)\,. \tag{4.106}$$

We will assume that $\mathscr{L}$ is meromorphic in π, and that $\mu^A\mathscr{L}_A$ and $\nu^A\mathscr{L}_B$ have simple poles at $\pi\sim\alpha_j$ and $\pi\sim\beta_j$ respectively for $j=1,\dots,n$. We can solve this equation at any π away from the simple poles by expressing $\mathscr{L}_A|_\pi$ in pure gauge. Doing so at γ_i for $i=1,\dots,n+2$ we have

$$\mathscr{L}_A|_{\pi\sim\gamma_i} = -\bar{\partial}_A\sigma_i\sigma_i^{-1}|_{\pi\sim\gamma_i} \tag{4.112}$$

where $\sigma:\mathbb{R}^4\to G^{n+2}$. This is enough to completely determine $\mathscr{L}$ in terms of σ. Indeed this reproduces equation (4.96).

The Lax equation is invariant under the standard gauge symmetry, $\mathscr{L}_A\mapsto h\mathscr{L}_Ah^{-1}-\bar{\partial}_Ahh^{-1}$, for h independent of π, which we have seen acts on σ by mapping $\sigma_i\mapsto h\sigma_i$. Hence modulo this gauge symmetry σ takes values in $G\backslash G^{n+2}$.

Requiring that the lax equation holds to second order at the γ_i, which is achieved by taking the Lie derivative of (4.111) along

$$\xi_i = -\frac{\langle\pi\,\hat{\gamma}_i\rangle\hat{\gamma}_i^{A'}}{\|\gamma_i\|^2}\frac{\partial}{\partial\pi^{A'}} \tag{4.113}$$

and evaluating at $\pi\sim\gamma_i$, gives the equations of motion (4.110).

Conversely the conditions (4.112) and equations of motion (4.110) are sufficient to ensure that the Lax equation holds for all π. To see why, note that the left hand side of (4.111) is a meromorphic section of $\mathcal{O}(2)$ with simple poles at $\pi\sim\alpha_i,\beta_i$. This is a $2n+3$ dimensional space. The conditions (4.112) ensure that it vanishes to first order at the γ_i, and the conditions (4.110) extend this to second order. This gives $2n+4$ constraints in total, with one lost to the gauge symmetry. The Lax equation follows.

Unfortunately we have been unable to give a straightforward interpretation to the Lax equation on spacetime. It is our expectation that this theory, and similar theories obtained from HCS on $\mathbb{PT}$ for choices of Ω which have zeros, will be in appropriate sense integrable.

Remark. We observe in chapter 5 that a 2d integrable theory of coupled σ-models arises as a symmetry reduction of the one considered here. This is notable because it has not been obtained as a symmetry reduction of the ASDYM equations.

4.5 Reality conditions

In this section we generalise the results of the preceding chapter to Lorentzian and ultrahyperbolic signatures. This is important because many lower dimensional integrable systems are known to arise from reductions of the ASDYM equations only in ultrahyperbolic signature. We also discuss the related issue of how to restrict the gauge group to a real form.

4.5.1 Lorentzian and ultrahyperbolic signatures

So far we have studied HCS on the twistor space of Euclidean spacetime. At first glance the theory does not straightforwardly generalize to Lorentzian and hyperbolic signatures. For example, the twistor space of Minkowksi space has only 5 real dimensions, and does not fibre over spacetime. Instead of attempting to define HCS theory on twistor space, it is more fruitful to realise it on the left-handed projective spin bundle. In Euclidean signature this coincides with $\mathbb{PT}$, so this is no different. In other signatures, the projective spin bundle fibres over both spacetime $\mathcal{M}$ and its associated twistor space $\mathcal{PT}$. In this context, it is often called the correspondence space and we will denote it by $\mathcal{F}$. As a smooth manifold $\mathcal{F} \cong \mathcal{M} \times \mathbb{CP}^1$. A review of the twistor correspondence in arbitrary signature is included in appendix B.6.

Apart from in Euclidean signature, the correspondence space is not naturally a complex manifold. To overcome this, first note that HCS on $\mathbb{PT}$ is equivalent to the following action

$$\frac{1}{2\pi i}\int_{\mathbb{PT}} \Omega \wedge \mathrm{CS}(\mathcal{A})\,, \tag{4.114}$$

where $\mathrm{CS}(\mathcal{A})$ is the full CS 3-form constructed using a full connection 1-form $\mathcal{A}$. Viewing the dynamical field as a full connection means that, in addition to the standard gauge invariance, the action (4.114) has a new, rather trivial, redundancy

$$\mathcal{A} \mapsto \mathcal{A} + e^0 \delta\mathcal{A}_0 + e^A \delta\mathcal{A}_A \tag{4.115}$$

This new gauge freedom can fixed by requiring that $\mathcal{A}$ is a partial connection, and doing so recovers the standard HCS action. From this perspective it's clear that the theory is only sensitive to the complex structure through the choice of $(3,0)$-form Ω.

Fortunately, there is a natural weighted 3-form on the projective spin bundle over spacetime of any signature. Let $\mathcal{PT} \overset{\rho}{\leftarrow} \mathcal{F} \overset{\pi}{\rightarrow} \mathcal{M}$ be the twistor correspondence for a real form $\mathcal{M}$ of complexified spacetime $\mathbb{CM}^4 = \mathbb{C}^4$. We can pullback the $(3,0)$-form $\mathrm{D}^3 Z$ which has holomorphic weight 4 from $\mathbb{PT}$, which we recall is the twistor space of complexified spacetime, by the embedding $\iota : \mathcal{PT} \hookrightarrow \mathbb{PT}$. Pulling back again by

$\rho : \mathcal{F} \to \mathcal{PT}$ to the correspondence space gives

$$(\iota \circ \rho)^* \, \mathrm{D}^3 Z = \frac{e^0 \wedge e^A \wedge e_A}{2} \, . \tag{4.116}$$

Here we are using coordinates (x, π) on $\mathcal{F}$, and have defined e^0, e^A in exactly the same way as they were defined in Euclidean signature (4.7). This is possible because, in contrast to $\bar{e}^0, \bar{e}^A$, spinor conjugation is not involved their definition. We can therefore use the action (4.114) in both Lorentzian and hyperbolic signatures with essentially no modification by treating it as an action on the correspondence space

$$\frac{1}{2\pi i} \int_{\mathcal{F}} \Omega \wedge \mathrm{CS}(\mathcal{A}) \, , \tag{4.117}$$

with $\Omega = (\iota \circ \rho)^* (\mathrm{D}^3 Z \otimes \Phi)$ for Φ a meromorphic section of $\mathcal{O}(-4) \to \mathbb{PT}$.

It is instructive the compare this situation to that of CS_4. The similarity is uncanny. It too is defined on a space $\mathcal{V}$ which double fibres over a spacetime, Σ, and a complex manifold, C. Furthermore, its Lagrangian is also the wedge product of a top holomorphic form on this complex manifold pulled back to $\mathcal{V}$ with the CS 3-form. This similarity is not a coincidence, as we shall see in chapter 5.

In sections 4.3 & 4.4 we demonstrated that in a number of cases that HCS theory on $\mathbb{PT}$ had an effective description on $\mathbb{R}^4$. In fact none of these calculations were sensitive to our choice of real slice $\mathbb{R}^4 \subset \mathbb{CM}^4$. We could equally have started with the action (4.117) on $\mathcal{F}$ over any real form $\mathcal{M} \subset \mathbb{CM}^4$. This is reflected in the fact that all of the effective spacetime actions we obtain are independent of spinor conjugation in Euclidean signature, $\pi \mapsto \hat{\pi}$. This may be surprising, since we often made use of spinor conjugation in deriving effective spacetime actions, but in fact it was only ever used in fixing the gauge.

4.5.2 Real forms of the gauge group

So far we have understood how to obtain actions for ASDYM equations on spacetimes of arbitrary signature for a simple, complex gauge group G. We now show how to restrict the gauge group to a real form, $G_{\mathbb{R}}$.

Such real forms arise as the fixed point sets of an involutive automorphism, $\Theta : G \to G$, which is conjugate-linear on the Lie algebra $\mathfrak{g}$. We write $\theta : \mathfrak{g} \to \mathfrak{g}$ for the induced map on the Lie algebra. The simplest example is $G_{\mathbb{R}} = \mathrm{SU}_n \subset G = \mathrm{SL}_n(\mathbb{C})$, for which we may take

$$\Theta : U \mapsto (U^\dagger)^{-1}\,, \qquad \theta : X \mapsto -X^\dagger\,. \tag{4.118}$$

In Lorentzian signature spinor conjugation swaps left-handed and right-handed spinors, and so swaps the SD and ASD parts of the curvature. As such there are no ASD connections on Minkowski space for real gauge groups. We therefore restrict our attention to Euclidean and ultrahyperbolic signatures.

Euclidean signature

In Euclidean signature we can work with a partial connection $\bar{\mathcal{A}} \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$. Writing $C : (x, \pi) \mapsto (x, \hat{\pi})$ for spinor conjugation on $\mathbb{PT}$, we impose the reality condition

$$C^* \bar{\mathcal{A}} = \theta(\bar{\mathcal{A}})\,. \tag{4.119}$$

We emphasise that this equation makes sense for a partial connection $\bar{\mathcal{A}}$, since both sides of the above equation are $(1, 0)$-forms with values in $\mathfrak{g}$. For a gauge transformation with parameter g to preserve this constraint it must obey

$$C^* g = \Theta(g)\,. \tag{4.120}$$

Note the curious fact that, since C has no fixed points, at no point in $\mathbb{PT}$ are any components of $\bar{\mathcal{A}}$ required to take values in $\mathfrak{g}_{\mathbb{R}}$, or are gauge transformations required to take values in $G_{\mathbb{R}}$.

Using the identity

$$\overline{\mathrm{HCS}(\bar{\mathcal{A}})} = \overline{\mathrm{HCS}}(\theta(\bar{\mathcal{A}}))\,, \tag{4.121}$$

where $\overline{\mathrm{HCS}}$ is defined using ∂ instead of $\bar{\partial}$, and taking care to note that C is orientation reversing, we have

$$\begin{aligned} S = \frac{1}{2\pi i}\int \Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}}) &= -\frac{1}{2\pi i}\int C^*\big(\Omega \wedge \mathrm{HCS}(\bar{\mathcal{A}})\big) = -\frac{1}{2\pi i}\int (C^*\Omega) \wedge \overline{\mathrm{HCS}}(C^*\bar{\mathcal{A}}) \\ &= -\frac{1}{2\pi i}\int (C^*\Omega) \wedge \overline{\mathrm{HCS}}(\theta(\bar{\mathcal{A}})) = \overline{\frac{1}{2\pi i}\int \overline{(C^*\Omega)}\mathrm{HCS}(\bar{\mathcal{A}})}\,. \end{aligned} \tag{4.122}$$

Hence the action will be real if

$$\overline{C^*\Omega} = \Omega\,. \tag{4.123}$$

Of course, our boundary conditions on $\bar{\mathcal{A}}$ must also be consistent with the reality conditions (4.119).

Example 4.5.1. The simplest choice of Ω compatible with this constraint is a specialisation of the one proposed by Costello and discussed in section 4.3.1

$$\Omega = \frac{e^0 \wedge e^A \wedge e_A}{2\langle \pi\, \alpha\rangle^2 \langle \pi\, \hat{\alpha}\rangle^2}\,, \tag{4.124}$$

where we have fixed $\beta = \hat{\alpha}$. Our boundary conditions are $\bar{\mathcal{A}}|_{\pi\sim\alpha} = \bar{\mathcal{A}}|_{\pi\sim\hat{\alpha}} = 0$.

We proceed in the familiar way by fixing the gauge using the frame field $\hat{\sigma} : \mathbb{PT} \to G$. It has the usual redundancies $\hat{\sigma} \mapsto h\hat{\sigma}g^{-1}$. The holomorphic Wilson line from α to $\hat{\alpha}$,

$$\mathcal{W}_{\alpha\to\hat{\alpha}} = \sigma = \sigma_{\hat{\alpha}}^{-1}\sigma_\alpha\,, \tag{4.125}$$

exhausts the gauge invariant data that can be extracted from $\hat{\sigma}$. (For convenience we write $\sigma_\alpha = \hat{\sigma}|_{\pi\sim\alpha}$ and $\sigma_{\hat{\alpha}} = \hat{\sigma}|_{\pi\sim\hat{\alpha}}$.) Our reality conditions imply that the combination

$$(C^*\hat{\sigma})\Theta(\hat{\sigma}^{-1}) = \rho \tag{4.126}$$

is independent of π, but can in principle depend on x. ρ obeys

$$\rho = C^*\rho = \hat{\sigma}\Theta(C^*\hat{\sigma}^{-1}) = (\Theta(C^*\hat{\sigma})\hat{\sigma}^{-1})^{-1} = \Theta(\rho^{-1})\,. \tag{4.127}$$

It's invariant under the right action on $\hat{\sigma}$, but is not under the left action:

$$\rho \mapsto h\rho\Theta(h^{-1})\,. \tag{4.128}$$

We refer to this as congruency. Under the assumption that ρ lies in the same congruency class as the identity matrix, we may fix $\rho = \mathrm{id}.$. This does not completely fix the ambiguity in $\hat{\sigma}$: we may still act on the left by $h : \mathbb{R}^4 \to G_\mathbb{R}$, which may be interpreted as the gauge redundancy of the real ASD connection on spacetime. We cannot eliminate this residual symmetry by fixing the value $\hat{\sigma}$ at any point in $\mathbb{CP}^1_x$, since it need not take values in $G_\mathbb{R}$ anywhere. The holomorphic Wilson line from α to $\hat{\alpha}$ is

$$\sigma = \sigma_{\hat{\alpha}}^{-1}\sigma_\alpha = \Theta(\sigma_\alpha^{-1})\sigma_\alpha\,, \tag{4.129}$$

and obeys

$$\sigma = \Theta(\sigma^{-1})\,. \tag{4.130}$$

In particular σ does not take values in a Lie group. For example, if we take $G_{\mathbb{R}} = \mathrm{SU}_n$, then σ is a Hermitian form. This rather strange result for the reality condition on Yang's matrix has been observed elsewhere [176, 54].

The resulting spacetime action is just that of WZW$_4$ (4.39) taking $\sigma = \Theta(\sigma_\alpha^{-1})\sigma_\alpha$ as its argument. This is obviously invariant under $\sigma_\alpha \mapsto h\sigma_\alpha$ for $h : \mathbb{R}^4 \to G_{\mathbb{R}}$, and so should be viewed as a σ-model on the coset space $G_{\mathbb{R}}\backslash G$. In subsection 4.3.1 we observed that for $\beta = \hat{\alpha}$ the self-dual 2-form $\omega_{\alpha,\beta}$ appearing in the action of WZW$_4$ is proportional to the Kähler form in the complex structure determined by α. Our treatment here now justifies this choice.

The Lax connection is determined by a spacetime gauge field $\bar{\mathcal{A}}'_A = A_{AA'}\pi^{A'}$, and our boundary conditions imply

$$A_{AA'} = \alpha_{A'}\hat{\alpha}^{B'}\partial_{AB'}\Theta(\sigma_\alpha)\Theta(\sigma_\alpha^{-1}) - \hat{\alpha}_{A'}\alpha^{B'}\partial_{AB'}\sigma_\alpha\sigma_\alpha^{-1}\,. \tag{4.131}$$

It is clear that $A = \theta(A)$, and that the left action $\sigma_\alpha \mapsto h\sigma_\alpha$ really does correspond to a spacetime $G_{\mathbb{R}}$ gauge symmetry.

Unfortunately the constraint $\Omega = \overline{C^*\Omega}$ is rather restrictive. Most disappointingly, it prevents us making the choice $\Omega = \mathrm{D}^3 Z/\langle\pi\,\alpha\rangle^4$ which was used to generate the LMP action in subsection 4.3.2.

Ultrahyperbolic signature

We now turn our attention to the case of ultrahyperbolic $(++--)$ signature spacetime. The reality conditions are essentially identical to those used in the Euclidean case, and are also closely related to the corresponding conditions imposed in CS$_4$ [59].

Writing $C : (x,\pi) \mapsto (x,\bar{\pi})$ for spinor conjugation, where $\bar{\pi}$ is the component-wise complex conjugate of π, we require that the gauge field $\mathcal{A}$ on the correspondence space obeys

$$C^*\mathcal{A} = \theta(\mathcal{A})\,. \tag{4.132}$$

This will be compatible with the gauge redundancies

$$\mathcal{A} \mapsto \mathcal{A} + e^0\delta\mathcal{A}_0 + e^A\delta\mathcal{A}_A\,, \qquad \mathcal{A} \mapsto g\mathcal{A}g^{-1} - \mathrm{d}gg^{-1} \tag{4.133}$$

provided the gauge parameters likewise obey $C^*\delta\mathcal{A}_0 = \theta(\delta\mathcal{A}_0)$, $C^*\delta\mathcal{A}_A = \theta(\delta\mathcal{A}_A)$, and $C^*g = \Theta(g)$. The action (4.117) on $\mathcal{F}$ will then be real if in addition

$$\Omega = \overline{C^*\Omega}\,. \tag{4.134}$$

This constraint is far easier to satisfy in ultrahyperbolic as compared to Euclidean signature, since spinor conjugation fixes a circle in $\mathbb{CP}^1$. For example, we could simply assume that all poles and zeros of Ω lie on this real circle. In particular, unlike in Euclidean signature, it is now possible to obtain a real LMP action by taking $\Omega = e^0 \wedge e^A \wedge e_A/2\langle\pi\,\alpha\rangle^4$ with $\alpha = \bar{\alpha}$. It is easy to see that the resulting effective spacetime action in this case is the LMP action (4.66) with ϕ taking values in $\mathfrak{g}_{\mathbb{R}}$. The infinitesimal conformal symmetries (4.68) & (4.69) of the LMP Lagrangian can be consistently realised on ultrahyperbolic spacetime.

It also instructive to reconsider $\Omega = e^0 \wedge e^A \wedge e_A/2\langle\pi\,\alpha\rangle^2\langle\pi\,\beta\rangle^2$. If α, β are real spinors then the holomorphic Wilson line from α to β takes values in $G_{\mathbb{R}}$. This descends to WZW_4 with σ taking values in $G_{\mathbb{R}}$. Alternatively we can take $\beta = \bar{\alpha}$, which is similar to the Euclidean case. It leads to the action of WZW_4 with argument $\sigma = \Theta(\sigma_\alpha^{-1})\sigma_\alpha$ for some $\sigma_\alpha : \mathbb{R}^4 \to G_{\mathbb{R}}\backslash G$.

Finally, in the case where we impose trigonometric boundary conditions at pairs of simple poles in Ω the appropriate reality conditions are a little more involved. If the two simple poles both occur at real spinors, then we choose $\mathfrak{g}_\pm$ to be Lagrangian subalgebras of $\mathfrak{g}_{\mathbb{R}}$, so that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_-, \mathfrak{g}_+)$ is a real Manin triple. In the case where the simple poles occur at conjugate spinors it is natural to relax our boundary conditions somewhat so that the residues generated at the two poles cancel one another. For further details we refer the reader to [60], where analogous boundary conditions were introduced in CS_4.

Whilst ultrahyperbolic signature is of less physical interest in 4d, it is often more convenient as a starting point for generating 2d integrable theories via symmetry reduction. This is because of the greater flexibility it offers in reductions and reality conditions [124].

4.6 Discussion

In this chapter we have seen how a number of 4d field theories with classical equations of motion equivalent to the ASDYM equations can be obtained from HCS theory on twistor space defined with a meromorphic (3,0)-form Ω. We have also seen that for a choice of Ω with zeros we obtain a 4d theory on spacetime which has equations of motion which are not equivalent to the ASDYM equations, but nonetheless admits a 4d analogue of a Lax connection.

There are a number of natural extensions to this work. First and foremost among them is to better understand the integrability of the 4d theories we have derived. By analogy with 2d CIFT it is to be hoped that the Lax connection can be used to generate an infinite family of conserved charges. One natural guess for how this might work is by forming a family of holomorphic Wilson lines from the Lax in the complex structure determined by π. Expanding in a power series around some $\pi \sim \pi_0$ at which $\mathscr{L} = \bar{\mathcal{A}}'$ vanishes should then generate an infinite set of non-local conserved quantities. In 2d CIFT there are known sufficient conditions which guarantee that an infinite subset of these are in involution with one another [114, 115]. It would be interesting to see if similar results could be deduced in the 4d case.

Another intriguing possibility is to explore gravitational analogues of this construction. ASD $\mathcal{N} = 4$ superconformal gravity and ASD $\mathcal{N} = 8$ Einstein supergravity have twistor descriptions as Kodaira-Spencer and Poisson CS theory respectively [28, 123]. Bosonic analogues of both could be obtained by removing supersymmetry and introducing poles on twistor space.

But by far the most interesting direction is to study these theories in the quantum setting. Costello and Li have recently shown that HCS suffers from a series of anomalies, but by coupling to Kodaira-Spencer gravity these can be eliminated for particular choices of the gauge group [48, 47]. (The only bosonic candidate is $G = \mathrm{SO}(8)$.) This also leads to a unique quantization of the theory despite its apparent non-renormalizability. We hope that this might lead to 4d counterparts of 2d QIFTs, and perhaps analogues of the YBE and Yangian.

Chapter 5

Symmetry reductions

In this chapter we relate two very different approaches to the study of classical integrable systems: namely as symmetry reductions of the ASDYM equations and in terms of CS_4.

On the one hand, many integrable PDEs are known to arise as symmetry reductions of the ASDYM equations [124]. These include, but are not limited to, the Bogomolny [87], Ernst [171, 161], non-linear Schrödinger [122], Toda field and chiral equations [162]. To perform a symmetry reduction one fixes a subgroup of the conformal group, lifts it to act on the adjoint bundle of the gauge theory and then requires that the connection be invariant under its action. The ASDYM equations then simplify, in many cases leading to a lower dimensional integrable PDE. Indeed, Ward conjectured that all integrable PDEs can be generated by this procedure [162], although this is no longer believed to be true. (A notable integrable system that is not known to arise as a symmetry reduction of the ASDYM equations is the Kadomtsev-Petviashvili equation [98, 124].[1])

On the other, there has been much recent progress in describing 2d CIFTs using CS_4 [53, 60, 25, 102]. This is achieved by solving for the partial connection in the presence of order and disorder defects. We reviewed how this works in the case of the PCM in subsection 1.3.6. Some of the integrable systems obtained in this way have equations of motion which are not known to arise as symmetry reductions of the ASDYM equations, for example, the integrable coupled σ-models of [58, 59].

In this chapter we argue that these two apparently very different approaches are related by starting with HCS on twistor space, or more generally CS_6 on twistor correspondence space. We have seen in chapter 4 that, for meromorphic choices of the $(3,0)$-form Ω, HCS describes integrable theories on 4d spacetime, many of which

[1] It can be obtained as a symmetry reduction for an infinite dimensional gauge group [2].

have classical equations of motion equivalent to the ASDYM equations. Performing a symmetry reduction by an appropriate subgroup of the conformal group, these 4d actions descend to actions for lower dimensional integrable systems.

On the other hand, if we instead lift this conformal group to the correspondence space and perform the symmetry reduction there we obtain a mixed holomorphic/topological CS theory on the quotient. This describes the lower dimensional integrable spacetime theory.

If the orbits of the chosen subgroup of the conformal group are 2d dimensional, then we arrive at a CS_4 description of a 2d CIFT. This framework is illustrated in the diagram below:

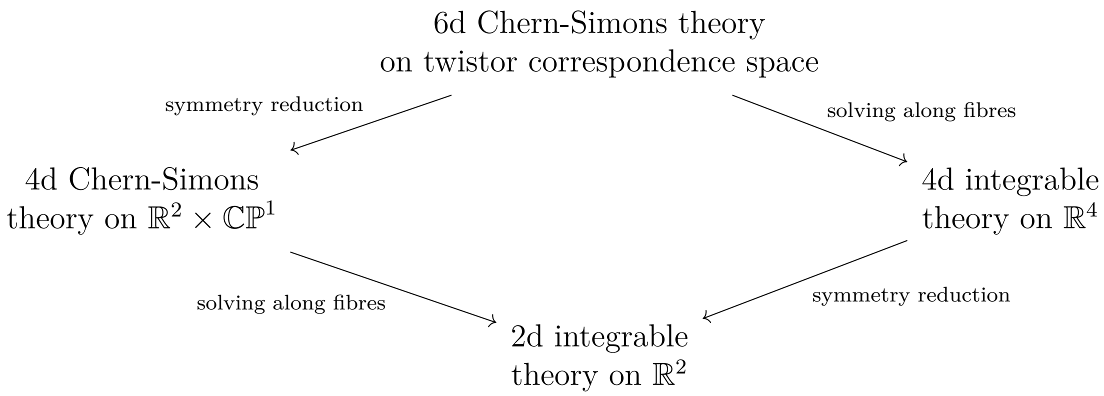


Fig. 5.1 A guide to the relationship between CS type theories and integrable systems in dimensions 2 and 4

In this chapter we explore this set up in the simplest case, where we perform a symmetry reduction by a 2 dimensional group of translations. (We further require that the induced metric on the orbits be non-degenerate.)

We also consider performing a symmetry reduction by a 1 dimensional group of non-null translations. This leads to a 5d partially holomorphic CS theory on minitwistor correspondence space describing the Bogomolony equations.

5.1 From CS_6 to CS_4

In this section we demonstrate the commutativity of the figure 5.1 in the simplest possible case. In particular, we start with CS_6 theory on twistor correspondence space in ultrahyperbolic signature with

$$\Omega = \frac{e^0 \wedge e^A \wedge e_A}{2\langle\pi\,\alpha\rangle^2\langle\pi\,\beta\rangle^2}\,. \tag{5.1}$$

We know from chapter 4 that this descends to WZW_4 on $\mathbb{R}^{2,2}$. Here we show that performing a symmetry reduction by a 2d group of translations on spacetime leads to the PCM, and that lifting this group to correspondence space and performing the reduction there leads to its description in CS_4. At the end of the section we consider more general choices for Ω.

We assume familiarity with the background material reviewed in subsection 1.3.6.

5.1.1 PCM as a symmetry reduction of $\mathbf{WZW_4}$

We begin by verifying that the symmetry reduction of WZW_4 on spacetime by a 2d group of translations is the PCM. This is clear at the level of the classical equations of motion [124]. We work in utlrahyperbolic signature for the greater freedom it provides in reality conditions.

Let $\mathcal{H}^{--}$ be the 2d group of translations generated by $Y+\bar{Y}$ and $i(Y-\bar{Y})$ for

$$Y = \kappa^A\gamma^{A'}\partial_{AA'}\,, \qquad \bar{Y} = \bar{\kappa}^A\bar{\gamma}^{A'}\partial_{AA'}\,. \tag{5.2}$$

We further assume without loss of generality that $[\kappa\,\bar{\kappa}] = \langle\gamma\,\bar{\gamma}\rangle = i$, and hence that $\eta(Y,\bar{Y}) = -1$ for η the standard flat metric with signature $(++--)$ on $\mathbb{R}^{2,2}$. The orbits of $\mathcal{H}^{--}$ are 2-planes on which the pullback of η is non-degenerate and negative definite. Each intersects the level set $\Sigma = \{x^{AA'}\bar{\kappa}_A\bar{\gamma}_{A'} = x^{AA'}\kappa_A\gamma_{A'} = 0\} \subset \mathbb{R}^{2,2}$ once, allowing us to identify Σ with the quotient $\mathcal{H}^{--}\backslash\mathbb{R}^{2,2}$. We introduce natural coordinates on this subspace

$$w = x^{AA'}\bar{\kappa}_A\gamma_{A'}\,, \qquad \bar{w} = x^{AA'}\kappa_A\bar{\gamma}_{A'}\,. \tag{5.3}$$

To implement the reduction we impose invariance under Y and $\bar{Y}$ directly on Yang's matrix

$$\mathcal{L}_Y\sigma = \mathcal{L}_{\bar{Y}}\sigma = 0\,. \tag{5.4}$$

The action is then clearly divergent, which can be attributed to the fact that $\mathcal{H}^{--}$ is non-compact. To overcome this issue we should compactify $\mathbb{R}^{2,2}$ to $\mathbb{R}^2\times\mathrm{T}^2$, but to avoid the geometric complications that this would introduce we will simply discard the divergent factor of $\mathrm{Vol}\,\mathcal{H}^{--}$. In practice we contract the bivector $iY\wedge\bar{Y}$ into the

Lagrangian (viewed as a top form) of WZW_4, which has the effect of saturating the form components in the invariant directions. Under the assumed translation invariance (5.4) the resulting 2-form can then be pushed forward to the quotient space, or equivalently pulled back via the embedding $\iota_\Sigma : \Sigma \to \mathbb{R}^{2,2}$.

Recall the action of WZW_4 (4.39). We take the spinors α, β to be real with $\langle \alpha\, \beta \rangle = 1$. Reducing the two terms in the Lagrangian separately gives

$$\begin{aligned} \iota_\Sigma^*\Big((iY \wedge \bar{Y}) \lrcorner \operatorname{tr}(J \wedge *J) \Big) &= \operatorname{tr}(J_\Sigma \wedge *_\Sigma J_\Sigma)\,, \\ \iota_\Sigma^*\Big((iY \wedge \bar{Y}) \lrcorner \operatorname{tr}(\omega_{\alpha,\beta} \wedge \tilde{J}^3) \Big) &= k \operatorname{tr}(\tilde{J}_\Sigma^3)\,, \end{aligned} \tag{5.5}$$

where $k = (iY \wedge \bar{Y}) \lrcorner \omega_{\alpha,\beta} = \langle \alpha\, \gamma \rangle \langle \beta\, \bar{\gamma} \rangle + \langle \alpha\, \bar{\gamma} \rangle \langle \beta\, \gamma \rangle$ and $*_\Sigma$ denotes the Hodge star for the standard flat metric δ_Σ on Σ. We are also writing $\sigma_\Sigma = \iota_\Sigma^* \sigma$, and have defined J_Σ and $\tilde{J}_\Sigma$ in the usual way. Therefore the symmetry reduction of WZW_4 (4.39) by $\mathcal{H}^{--}$ is the PCM:

$$S_{\mathrm{PCM}}[\sigma_\Sigma] = \frac{1}{2} \int_\Sigma \operatorname{tr}(J_\Sigma \wedge *_\Sigma J_\Sigma) + \frac{k}{3} \int_{\Sigma \times [0,1]} \operatorname{tr}(\tilde{J}_\Sigma^3)\,. \tag{5.6}$$

Remark. Consider instead the 2d group of translations $\mathcal{H}^{+-}$ generated by

$$Y^+ = \kappa^A \gamma^{A'} \partial_{AA'}\,, \qquad Y^- = \lambda^A \delta^{A'} \partial_{AA'} \tag{5.7}$$

for $\gamma, \delta, \kappa, \lambda$ real spinors obeying $\langle \gamma\, \delta \rangle = [\kappa\, \lambda] = 1$. Note that $\eta(Y^+, Y^-) = 1$. The orbits of $\mathcal{H}^{+-}$ are 2-planes on which the pullback of η is non-degenerate and indefinite. The symmetry reduction of WZW_4 with α and β real is the PCM on $\mathbb{R}^{1,1}$. The coefficient of the WZW term is $k = (Y^- \wedge Y^+) \lrcorner \omega_{\alpha,\beta} = \langle \alpha\, \gamma \rangle \langle \beta\, \delta \rangle + \langle \alpha\, \delta \rangle \langle \beta\, \gamma \rangle$.

We could also apply either reduction to the case where α and β are not real, but conjugate. This would lead to a PCM with target the coset space $G_\mathbb{R} \backslash G$. This is also the result of performing the reduction in Euclidean signature.

5.1.2 CS_4 as a symmetry reduction of CS_6

In this section we perform the reduction by $\mathcal{H}^{--}$ on the correspondence space $\mathcal{F}$. (We may also think of this as the projective spinor bundle of $\mathbb{R}^{2,2}$.) Recall from section 4.5 that the 6d action describing WZW_4 is

$$S_{\mathrm{CS}_6}[\mathcal{A}] = \frac{1}{2\pi i} \int_\mathcal{F} \Omega \wedge \mathrm{CS}(\mathcal{A}) \tag{5.8}$$

for

$$\Omega = \frac{e^0 \wedge e^A \wedge e_A}{2\langle \pi\, \alpha\rangle^2 \langle \pi\, \beta\rangle^2}\,. \tag{5.9}$$

As above we take α, β to be real spinors obeying $\langle \alpha\, \beta\rangle = 1$. Our boundary conditions are that $\mathcal{A}$ should vanish to first order at $\pi \sim \alpha, \beta$.

To implement the reduction we must first lift Y and $\bar{Y}$ to vector fields $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ on $\mathcal{F}$. These are required to obey

$$\pi_* \mathcal{Y} = Y\,, \qquad \pi_* \bar{\mathcal{Y}} = \bar{Y} \tag{5.10}$$

and preserve Ω. For translations on $\mathbb{R}^{2,2}$ these lifts are trivial

$$\mathcal{Y} = \kappa^A \gamma^{A'} \partial_{AA'}\,, \qquad \bar{\mathcal{Y}} = \bar{\kappa}^A \bar{\gamma}^{A'} \partial_{AA'}\,. \tag{5.11}$$

The quotient $\mathcal{H}^{--}\backslash\mathcal{F}$ can be identified with $\mathcal{V} = \iota_\Sigma^* \mathcal{F}$. We write $\iota_\mathcal{V} : \mathcal{V} \to \mathcal{F}$ for the obvious inclusion. As a smooth manifold $\mathcal{V} = \Sigma \times \mathbb{CP}^1$, and we use coordinates $(w, \bar{w}, [\pi]) = (x^{AA'}\bar{\kappa}_A \gamma_{A'}, x^{AA'}\kappa_A \bar{\gamma}_{A'}, [\pi])$ adapted to this splitting.

To perform the reduction we impose invariance under $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ directly on the gauge field $\mathcal{A}$, that is we demand

$$\mathcal{L}_\mathcal{Y} \mathcal{A} = \mathcal{L}_{\bar{\mathcal{Y}}} \mathcal{A} = 0\,. \tag{5.12}$$

The Lie derivative here is defined by lifting the action of $\mathcal{H}^{--}$ to the adjoint bundle. By working in a frame which is invariant under this lift, the Lie derivatives act as they would on 1-forms. The residual gauge freedom consists of gauge transformations which are invariant under $\mathcal{Y}$ and $\bar{\mathcal{Y}}$. As observed in [124], this explains the dependence of symmetry reductions of the ASDYM equations on the gauge in which they are expressed. At the level of the twistor action the gauge is essentially fixed by the boundary conditions.

Under these assumptions the Lagrangian of the theory is invariant under the action of $\mathcal{H}^{--}$, and so, as in subsection 5.1.1, the action is divergent. To eliminate this divergence we should compactify in the invariant directions, but instead we simply discard the overall factor of $\mathrm{Vol}\,\mathcal{H}^{--}$. In practice this is achieved by contracting the bivector $i\mathcal{Y} \wedge \bar{\mathcal{Y}}$ into the Lagrangian (viewed as a top form) of the theory whilst assuming (5.12). Instead of pushing forward to the quotient we can equivalently pull back by $\iota_\mathcal{V} : \mathcal{V} \to \mathcal{F}$.

Before performing the contraction with $i\mathcal{Y}\wedge\bar{\mathcal{Y}}$ it's useful to eliminate some of the redundancy in $\mathcal{A}$. Let us split $\mathcal{A}$ according to the smooth identification $\mathcal{F}\cong\mathbb{R}^{2,2}\times\mathbb{CP}^1$:

$$\mathcal{A}=\mathrm{d}x^{AA'}\mathcal{A}_{AA'}+\mathcal{A}_{\mathbb{CP}^1}\,. \tag{5.13}$$

We can use the ambiguity $\mathcal{A}\mapsto\mathcal{A}+e^A\delta\mathcal{A}_A$ to fix the 'gauge' $\mathcal{Y}\lrcorner\mathcal{A}=\bar{\mathcal{Y}}\lrcorner\mathcal{A}=0$. This requires choosing $\delta\mathcal{A}$ so that

$$\mathcal{Y}\lrcorner\mathcal{A}\mapsto\mathcal{Y}\lrcorner\mathcal{A}+\langle\pi\,\gamma\rangle[\delta\mathcal{A}\,\kappa]=0\,,\qquad\bar{\mathcal{Y}}\lrcorner\mathcal{A}\mapsto\bar{\mathcal{Y}}\lrcorner\mathcal{A}+\langle\pi\,\bar{\gamma}\rangle[\delta\mathcal{A}\,\bar{\kappa}]=0\,, \tag{5.14}$$

which implies

$$\delta\mathcal{A}_A=i\bigg(\frac{\kappa_A\bar{\kappa}^B\bar{\gamma}^{B'}}{\langle\pi\,\bar{\gamma}\rangle}-\frac{\bar{\kappa}_A\kappa^B\gamma^{B'}}{\langle\pi\,\gamma\rangle}\bigg)\mathcal{A}_{BB'}\,. \tag{5.15}$$

This leads to

$$\mathcal{A}\mapsto\mathcal{A}_{\mathbb{CP}^1}+i\,\mathrm{d}x^{AA'}\bigg(\frac{\kappa_A\bar{\gamma}_{A'}\bar{\kappa}^B\pi^{B'}}{\langle\pi\,\bar{\gamma}\rangle}-\frac{\bar{\kappa}_A\gamma_{A'}\kappa^B\pi^{B'}}{\langle\pi\,\gamma\rangle}\bigg)\mathcal{A}_{BB'}=\mathcal{A}_{\mathbb{CP}^1}+\mathrm{d}w\mathcal{A}_w+\mathrm{d}\bar{w}\mathcal{A}_{\bar{w}} \tag{5.16}$$

for

$$\mathcal{A}_w=-\frac{i\kappa^B\pi^{B'}\mathcal{A}_{BB'}}{\langle\pi\,\gamma\rangle}\,,\qquad\mathcal{A}_{\bar{w}}=\frac{i\bar{\kappa}^B\pi^{B'}\mathcal{A}_{BB'}}{\langle\pi\,\bar{\gamma}\rangle}\,. \tag{5.17}$$

We can see that imposing $\mathcal{Y}\lrcorner\mathcal{A}=\bar{\mathcal{Y}}\lrcorner\mathcal{A}=0$ necessitates introducing simple poles in $\mathcal{A}_w$ and $\mathcal{A}_{\bar{w}}$ at $\pi\sim\gamma$ and $\bar{\gamma}$ respectively.

The advantage of working in this 'gauge' is that when performing the symmetry reduction the bivector $i\mathcal{Y}\wedge\bar{\mathcal{Y}}$ may only contract into Ω, which gives

$$(i\mathcal{Y}\wedge\bar{\mathcal{Y}})\lrcorner\Omega=\frac{\langle\pi\,\gamma\rangle\langle\pi\,\bar{\gamma}\rangle}{\langle\pi\,\alpha\rangle^2\langle\pi\,\beta\rangle^2}e^0\,. \tag{5.18}$$

Writing $A=\iota_{\mathcal{V}}^*\mathcal{A}$, $\omega=\iota_{\mathcal{V}}^*(i\mathcal{Y}\wedge\bar{\mathcal{Y}}\lrcorner\Omega)$, the result of the symmetry reduction is

$$S_{\mathrm{CS4}}[A]=\frac{1}{2\pi i}\int_{\mathcal{V}}\omega\wedge\mathrm{CS}(A)\,, \tag{5.19}$$

the action of CS_4. Introducing the inhomogeneous coordinate $z=\langle\pi\,\alpha\rangle/\langle\pi\,\beta\rangle$ we have

$$\omega=\frac{(z-z_0)(z-\bar{z}_0)\,\mathrm{d}z}{i(\bar{z}_0-z_0)z^2} \tag{5.20}$$

for $z_0=\langle\gamma\,\alpha\rangle/\langle\gamma\,\beta\rangle$. The boundary conditions are that A should vanish to first order at $z=0,\infty$. From equation (5.17) we can see that A_w and $A_{\bar{w}}$ have simple poles at

$z = z_0$ and $\bar{z}_0$ respectively. It is clear that gauge transformations on $\mathcal{F}$ preserving the conditions (5.12) pull back to the standard gauge transformations of A. The requirement that they are trivial at $\pi \sim \alpha, \beta$ implies that their pullback is trivial at $z = 0, \infty$. The induced reality conditions are $C^* A = \theta(A)$ where $C : z \mapsto \bar{z}$. These are the standard reality conditions for CS_4 introduced in [60].

We have seen in subsection 1.3.6 that for this choice of ω and boundary conditions CS_4 describes a PCM on Σ

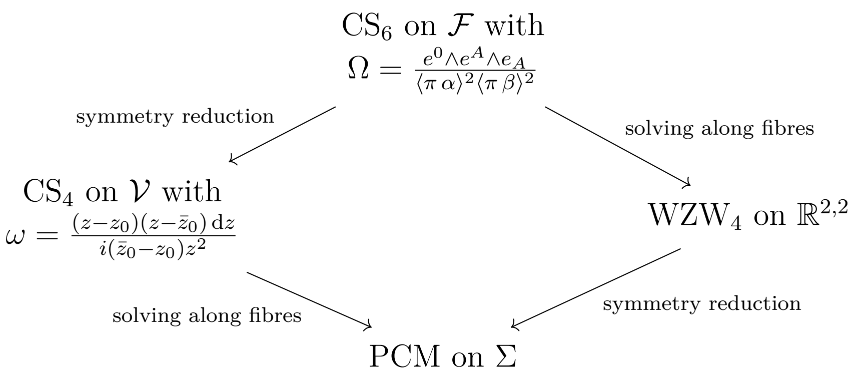
$$S_{\mathrm{PCM}}[\sigma_\Sigma] = \frac{1}{2}\int_\Sigma \mathrm{tr}(J_\Sigma \wedge *_\Sigma J_\Sigma) + \frac{k}{3}\int_{\Sigma\times[0,1]} \mathrm{tr}(\tilde{J}_\Sigma^3) \tag{5.21}$$

with

$$k = \frac{z_0 + \bar{z}_0}{i(\bar{z}_0 - z_0)} = \langle \alpha\, \gamma\rangle\langle \beta\, \bar{\gamma}\rangle + \langle \alpha\, \bar{\gamma}\rangle\langle \beta\, \gamma\rangle\,. \tag{5.22}$$

This is exactly the action we obtained in subsection 5.1.1 by performing the symmetry reduction directly on spacetime.

We therefore have the following commutative diagram:



Using the equations (5.17), the Lax connection $\mathcal{L}_A = -\langle \pi\,\beta\rangle \alpha^{A'}\sigma^{-1}\partial_{AA'}\sigma$ of WZW_4 descends to that of the PCM:

$$\mathcal{L} = \langle \pi\,\beta\rangle\left(\frac{\langle\alpha\,\gamma\rangle}{\langle\pi\,\gamma\rangle}J_\Sigma^{1,0} + \frac{\langle\alpha\,\bar{\gamma}\rangle}{\langle\pi\,\bar{\gamma}\rangle}J_\Sigma^{0,1}\right) = \frac{z_0}{z_0 - z}J_\Sigma^{0,1} + \frac{\bar{z}_0}{\bar{z}_0 - z}J_\Sigma^{1,0}\,. \tag{5.23}$$

To recover the Lax appearing in equation (1.122) we map $z \mapsto -1/z$, $z_0 \mapsto -1/z_0$.[2]

[2]This is necessary because in subsection 1.3.6 we defined σ_Σ to be the holomorphic Wilson line from $z = \infty$ to 0, whereas in subsection 4.3.1 we defined σ to be the holomorphic Wilson line from $\pi \sim \alpha$ to β. Our inhomogeneous coordinate on $\mathbb{CP}^1$ has then been chosen so that $\alpha \mapsto 0$, $\beta \mapsto \infty$. Correcting for this doesn't introduce a relative sign in the WZW term because we also need to reverse the direction of the homotopy $\tilde{\sigma}_\Sigma$.

Remark. We could instead have performed the reduction along $\mathcal{H}^{+-}$ introduced in equation (5.7). The argument proceeds in an identical way, and leads to the CS_4 description of the Lorentzian PCM [60]. Similarly, choosing α, β conjugate instead of real gives CS_4 descriptions of 2d σ-models on the coset $G_{\mathbb{R}}\backslash G$.

5.1.3 Symmetry reductions for different choices of Ω

The calculation in subsection 5.1.2 is not sensitive to the particular choice of Ω. If we quotient CS_6 on the correspondence space $\mathcal{F}$ with meromorphic $(3,0)$-form

$$\Omega = \Phi \frac{e^0 \wedge e^A \wedge e_A}{2} \tag{5.24}$$

by the 2d group of translations $\mathcal{H}^{--}$ we will obtain CS_4 on $\mathcal{V}$ with measure

$$\omega = \langle \pi\, \gamma \rangle \langle \pi\, \bar{\gamma} \rangle \Phi e^0 \,. \tag{5.25}$$

The dynamical field A is related to $\mathcal{A}$ by the conditions

$$A_{\mathbb{CP}^1} = \iota_\Sigma^* \mathcal{A}_{\mathbb{CP}^1}\,, \qquad A_w = -\frac{i\kappa^B \pi^{B'} \iota_\Sigma^* \mathcal{A}_{BB'}}{\langle \pi\, \gamma \rangle}\,, \qquad \mathcal{A}_{\bar{w}} = \frac{i\bar{\kappa}^B \pi^{B'} \iota_\Sigma^* \mathcal{A}_{BB'}}{\langle \pi\, \bar{\gamma} \rangle}\,. \tag{5.26}$$

We now apply the symmetry reduction to the remaining choices of Ω appearing in chapter 4.

- Choosing
$$\Omega = \frac{e^0 \wedge e^A \wedge e_A}{2\langle \pi\, \alpha \rangle^4} \tag{5.27}$$
leads to the LMP action 4.3.2. Quotienting by $\mathcal{H}^{--}$ gives CS_4 with
$$\omega = -\frac{iz}{k^2(z-1)^4}\, \mathrm{d}z \tag{5.28}$$
where
$$z = \frac{\langle \pi\, \gamma \rangle \langle \alpha\, \bar{\gamma} \rangle}{\langle \pi\, \bar{\gamma} \rangle \langle \alpha\, \gamma \rangle} \tag{5.29}$$
and $k = \langle \alpha\, \gamma \rangle \langle \alpha\, \bar{\gamma} \rangle$. Prior to our work, CS_4 for choices of ω with poles of order greater than 2 in ω had not been studied, though they have since appeared in [102]. The method outlined in subsection 4.3.2 can equally be applied to CS_4,

and the above ω leads to

$$S[\phi] = \int_\Sigma \mathrm{tr}\left(\frac{1}{2}\,\mathrm{d}\phi \wedge * \mathrm{d}\phi + \frac{2k}{3}\phi\,\mathrm{d}\phi \wedge \mathrm{d}\phi\right). \tag{5.30}$$

This is known as the pseudo-dual of the PCM [55, 132, 178]. Its classical equations of motion are equivalent to those of the PCM. It is clear that it arises as a symmetry reduction of the LMP action (4.66).

- Instead choosing

$$\Omega = \frac{e^0 \wedge e^A \wedge e_A}{\langle \pi\, \alpha_+\rangle \langle \pi\, \alpha_-\rangle \langle \pi\, \beta_+\rangle \langle \pi\, \beta_-\rangle} \tag{5.31}$$

and imposing trigonometric boundary conditions on $\mathcal{A}$ at $\pi \sim \alpha_\pm, \beta_\pm$ gives the trigonometric action 4.3.3. Symmetry reduction then leads to

$$\omega \propto \frac{(z-z_0)(z-\bar{z}_0)}{z(z-z_+)(z-z_-)}\,\mathrm{d}z\,, \tag{5.32}$$

where $z = \langle \pi\, \alpha_-\rangle / \langle \pi\, \alpha_+\rangle$. $z_\pm, z_0, \bar{z}_0$ are the images of $\beta_\pm, \gamma, \bar{\gamma}$ under this map respectively. CS_4 for the above ω was considered in [53], and descends to

$$S[\sigma_\pm] \propto z_0 \int_\Sigma \mathrm{tr}\Big(J_+^{1,0} \wedge J_-^{0,1}\Big) - \bar{z}_0 \int_\Sigma \mathrm{tr}\Big(J_-^{1,0} \wedge J_+^{0,1}\Big)\,. \tag{5.33}$$

Here $\sigma_\pm : \Sigma \to G_\pm$, $J_\pm = -\,\mathrm{d}\sigma_\pm \sigma_\pm^{-1}$. We observed in subsection 4.3.3 that the spacetime action corresponding to Ω agrees off-shell with WZW_4, it is therefore unsurprising that we find the above coincides with the PCM. (Simply split $\sigma = \sigma_-^{-1}\sigma_+$ and apply the Polyakov-Weigmann identity [145].) We have included it regardless for the sake of completeness.

- Finally suppose we take

$$\Omega = \frac{e^0 \wedge e^A \wedge e_A \prod_{j=1}^{n} \langle \pi\, \alpha_j\rangle \langle \pi\, \bar{\alpha}_j\rangle}{2\prod_{i=1}^{n+2} \langle \pi\, \gamma_i\rangle^2}\,. \tag{5.34}$$

This is a specialisation of the Ω considered in section 4.4. We tolerate poles in $\kappa^A \bar{\mathcal{A}}_A$ and $\bar{\kappa}^A \bar{\mathcal{A}}_A$ at $\pi \sim \alpha_j$ and $\bar{\alpha}_j$ respectively. Performing a symmetry reduction by the group $\mathcal{H}^{--}$ generated by $Y + \bar{Y}$ and $i(Y - \bar{Y})$ for

$$Y = \kappa^A \alpha^{A'} \partial_{AA'}\,, \qquad \bar{Y} = \bar{\kappa}^A \bar{\alpha}^{A'} \partial_{AA'}\,, \tag{5.35}$$

we recover CS_4 with

$$\omega = \frac{\prod_{j=1}^{n+1}\langle\pi\,\alpha_j\rangle\langle\pi\,\bar{\alpha}_j\rangle}{\prod_{i=1}^{n+2}\langle\pi\,\gamma_i\rangle^2}\, e^0 \propto \frac{\prod_{j=1}^{n+1}(z-q_j)(z-\bar{q}_j)}{z^2\prod_{i=1}^{n}(z-p_i)^2}\,\mathrm{d}z\,. \tag{5.36}$$

Here we have identified $\alpha_{n+1} = \alpha$ and introduced the inhomogeneous coordinate $z = \langle\pi\,\gamma_{n+1}\rangle/\langle\pi\,\gamma_{n+2}\rangle$. $p_i, q_j, \bar{q}_j$ are the images of $\gamma_i, \alpha_j, \bar{\alpha}_j$ under this map, with $p_{n+1} = 0$ and $p_{n+2} = \infty$. The partial connection A is required to vanish to first order at $z = p_i$ for $i = 1, \dots n+2$, but we permit simple poles in A_w and $A_{\bar{w}}$ at $z = q_j$ and $\bar{q}_j$ respectively for $j = 1, \dots, n+1$. This was shown in [53] to describe the integrable coupled σ models of [58, 59]. It is straightforward to verify that this also arises as a symmetry reduction of the action (4.100). This example is particularly interesting, since it describes an integrable system which is not known to arise as a symmetry reduction of the ASDYM equations. Nonetheless, we have obtained it starting with CS_6 on the correspondence space.

5.2 CS_5 on minitwistor correspondence space

In this section we consider symmetry reductions by 1 dimensional groups of translations. The most interesting integrable system known to arise in this manner is the Bogomolny equation describing magnetic monopoles. We will find that it can be described by a 5d Chern-Simons theory (CS_5) on minitwistor correspondence space, $\mathbb{PN}$. Indeed, we have the following relationships between Chern-Simons type and integrable theories.

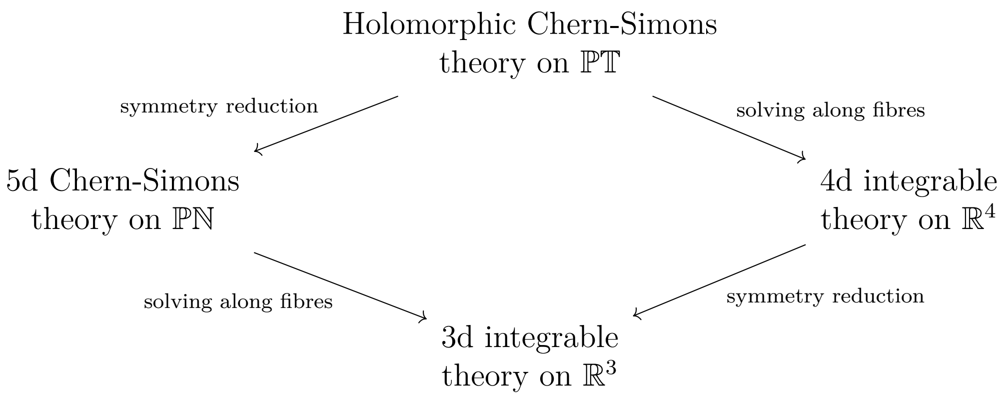


Fig. 5.2 A guide the relationship between CS type theories and integrable systems in dimensions 3 and 4

CS_5 on $\mathbb{PN}$ is a purely bosonic counterpart of the super minitwistor correspondence space action for the supersymmetric Bogomolny equations introduced in [146, 5].

5.2.1 Minitwistor correspondence

It is well known that the Bogomolny equations arise as a symmetry reduction of the ASDYM equations on 4d Euclidean spacetime by a 1 dimensional group of translations. In much the same way as ASD connections on 4d spacetime are described by holomorphic vector bundles on twistor space, solutions to the Bogomolny equations are described by holomorphic bundles over a complex manifold known as minitwistor space [160, 87]. This goes by the name of the Hitchin-Ward correspondence. In this subsection we review how minitwistor space arises as a quotient of $\mathbb{PT}$ by a translation.

Consider the quotient of $\mathbb{R}^4$ by the 1 dimensional group of translations $\mathcal{H}^+$ generated by the real vector X. The orbits of $\mathcal{H}^+$ each intersect the 3-plane $\mathbb{R}^3 \cong \{x^{AA'}X_{AA'} = 0\} \subset \mathbb{R}^4$ once, allowing us to identify it with the quotient $\mathcal{H}^+\backslash\mathbb{R}^4$. We write $\iota_{\mathbb{R}^3} : \mathbb{R}^3 \to \mathbb{R}^4$ for the embedding of this subspace into $\mathbb{R}^4$. It admits natural coordinates

$$y^{A'B'} = \varepsilon_{AB}x^{AA'}X^{BB'} , \tag{5.37}$$

where $y^{A'B'} = y^{B'A'}$. The choice of vector X breaks the $\mathrm{SO}_4(\mathbb{R})$ spacetime symmetry to $\mathrm{SO}_3(\mathbb{R})$, allowing us to identify primed and unprimed indices. For further details see appendix B.3. Assuming that $\delta(X, X) = 2$ the standard flat metric $\delta(y, y) = y^{A'B'}y_{A'B'}$ is induced on $\mathbb{R}^3$.

To perform this quotient on twistor space we must lift X to a vector field on $\mathbb{PT}$ which respects the complex structure. For a translation this lift is trivial: $\chi = X^{AA'}\partial_{AA'}$. Let $\mathbb{PN}$ be the quotient of $\mathbb{PT}$ by the translations generated by χ. $\mathbb{PN}$ can be identified with $\iota^*_{\mathbb{R}^3}\mathbb{PT}$, the pullback of the fibre bundle $\mathbb{PT} \to \mathbb{R}^4$ by the embedding $\iota_{\mathbb{R}^3}$. We write $\iota_{\mathbb{PN}} : \mathbb{PN} \hookrightarrow \mathbb{PT}$ for the natural inclusion. As a smooth manifold $\mathbb{PN} \cong \mathbb{R}^3 \times \mathbb{CP}^1$, and we use the coordinates $(y, [\pi])$ accordingly. It will be useful to introduce

$$n^{A'B'} = \frac{i(\pi^{A'}\hat{\pi}^{B'} + \hat{\pi}^{A'}\pi^{B'})}{\sqrt{2}\|\pi\|^2} , \tag{5.38}$$

a unit vector in $\mathbb{R}^3$ which is smoothly parametrised by $\mathbb{CP}^1$. Pushing forward the holomorphic structure by the quotient map induces a partially holomorphic structure on $\mathbb{PN}$. This is determined by the integrable subbundle locally generated by $\{\partial_0, \pi^{B'}\partial_{A'B'}\}$. Following [96, 95], we refer to to the partially holomorphic manifold $\mathbb{PN}$ as minitwistor correspondence space.[3]

[3] $\mathbb{PN}$ is also the space of light rays in four dimensional Minkowski space: the pair $(y, [\pi])$ uniquely determines a ray in $\mathbb{R}^{1,3}$ passing through the point y on a constant time slice in the direction $\pi^{A'}\bar{\pi}^{A}$. From this perspective, $\mathbb{PN}$ is the real codimension 1 slice $\{\bar{Z} \cdot Z = 0\} \subset \mathbb{PT}$, from which $\mathbb{PN}$ inherits its partially holomorphic structure as a CR manifold.

We can also exploit the complex structure on $\mathbb{PT}$ to take a different quotient. The vector field χ can be split as $\chi^{(1,0)} + \chi^{(0,1)}$ where

$$\chi^{(1,0)} = -\frac{X^{AA'}\pi_{A'}\hat{\pi}^{B'}\partial_{AB'}}{\|\pi\|^2}\,, \qquad \chi^{(0,1)} = \frac{X^{AA'}\hat{\pi}_{A'}\pi^{B'}\partial_{AB'}}{\|\pi\|^2} \tag{5.39}$$

take values in the holomorphic and antiholomorphic tangent bundles respectively. Together these vector fields generate an action of $\mathbb{C}$, the complexification of $\mathcal{H}^+$. The quotient of $\mathbb{PT}$ by this group is a complex manifold, which we refer to as minitwistor space $\mathbb{MT}$. It can be obtained as the quotient of minitwistor correspondence space by the pushforward of $i(\chi^{(0,1)} - \chi^{(1,0)})$, which is

$$\frac{i(\pi^{A'}\hat{\pi}^{B'} + \hat{\pi}^{A'}\pi^{B'})}{\|\pi\|^2}\partial_{A'B'} = \sqrt{2}n^{A'B'}\partial_{A'B'}\,. \tag{5.40}$$

This vector field generates translations in the direction n, and quotienting by it gives the space of oriented in lines in $\mathbb{R}^3$. We therefore have the minitwistor correspondence [87].

$$\begin{array}{ccccc} & & \mathbb{PN} & & \\ & \overset{\rho}{\swarrow} & & \overset{\pi}{\searrow} & \\ \mathbb{MT} & & & & \mathbb{R}^3 \end{array} \tag{5.41}$$

$\gamma = y^{A'B'}\pi_{A'}\pi_{B'}$ is invariant under the translation generated by $n^i\partial_i$, and so descends to the quotient. $[(\pi,\gamma)]$ are then homogeneous holomorphic coordinates on $\mathbb{MT}$, with $(\pi,\gamma) \sim (t\pi, t^2\gamma)$ for $t \in \mathbb{C}^*$. As a complex manifold $\mathbb{MT} \cong T^{1,0}\mathbb{CP} \cong \mathcal{O}(2) \to \mathbb{CP}^1$ with γ the holomorphic coordinate along the fibres. Explicitly $\rho : (x,[\pi]) \mapsto [(\gamma,\pi)] = [(y^{A'B'}\pi_{A'}\pi_{B'},\pi)]$.

As a partially holomorphic manifold $\mathbb{PN} \cong \mathbb{MT} \times \mathbb{R}$ with partial connection

$$\mathrm{d}' = \bar{\partial}_{\mathbb{MT}} + \mathrm{d}_{\mathbb{R}}\,. \tag{5.42}$$

5.2.2 CS$_5$ as a symmetry reduction of HCS

We can now apply the symmetry reduction by our 1 dimensional group of translations to HCS with measure $\Omega = \mathrm{D}^3 Z \otimes \Phi$. In the usual way we will not compactify in the χ direction, but instead simply discard the divergent integral. This is achieved by contracting the vector χ into the Lagrangian, which saturates the components in the invariant direction, and then pulling back by $\iota_{\mathbb{PN}} : \mathbb{PN} \to \mathbb{PT}$. We, of course, also assume that $\mathcal{L}_\chi \bar{\mathcal{A}} = 0$.

As in subsection 5.1.2, this is easiest to implement using the following trick. We replace the partial connection $\bar{\mathcal{A}}$ on $\mathbb{PT}$ with a full connection $\mathcal{A}$ which has a redundancy

$$\mathcal{A} \mapsto \mathcal{A} + e^0 \delta\mathcal{A}_0 + e^A \delta\mathcal{A}_A \,. \tag{5.43}$$

This ambiguity can be partially fixed by imposing the 'gauge' condition $\chi \lrcorner \mathcal{A} = 0$, which requires choosing

$$X^{AA'}(\mathcal{A}_{AA'} + \delta\mathcal{A}_A \pi_{A'}) = 0 \,. \tag{5.44}$$

Here we are writing $\mathcal{A} = \mathcal{A}_{AA'}\,\mathrm{d}x^{AA'} + \mathcal{A}_{\mathbb{CP}^1}$ as in subsection 5.1.2. One possible solution is

$$\delta\mathcal{A}_A = -\frac{\varepsilon_{AB} X^{BB'} \hat{\pi}_{B'}}{\|\pi\|^2} X^{CC'} \mathcal{A}_{CC'} \,, \tag{5.45}$$

which leads to

$$\mathcal{A} \mapsto \mathcal{A}_{\mathbb{CP}^1} + \mathrm{d}x^{AA'}(\mathcal{A}_{AA'} - X_{AA'} X^{BB'} \mathcal{A}_{BB'}) + \frac{i}{\sqrt{2}} \varepsilon_{AB} X^{BB'} n_{A'B'} X^{CC'} \mathcal{A}_{CC'} \,. \tag{5.46}$$

The advantage of working in this gauge is that when contracting the vector χ into the Lagrangian of HCS we find that it must hit Ω, giving

$$\chi \lrcorner \Omega = (e^0 \wedge e^\gamma) \otimes \Phi \tag{5.47}$$

for $e^\gamma = \mathrm{d}x^{A'B'} \pi_{A'} \pi_{B'}$. Pulling back by $\iota_{\mathbb{PN}}$ we therefore obtain CS_5 on minitwistor correspondence space

$$S_{\mathrm{CS}_5}[A] = \frac{1}{2\pi i} \int_{\mathbb{PN}} \omega \wedge \mathrm{CS}(A) \,. \tag{5.48}$$

Here $\omega = (e^0 \wedge e^\gamma) \otimes \Phi$ is the pullback by $\rho : \mathbb{PN} \to \mathbb{MT}$ of a meromorphic form on $\mathbb{MT}$. The field A is the pullback of the right hand side of equation (5.46) by $\iota_{\mathbb{PN}}$. Explicitly

$$A = \iota^*_{\mathbb{PN}} \mathcal{A} + \frac{i}{\sqrt{2}} \mathrm{d}y^{A'B'} n_{A'B'} \iota^*_{\mathbb{PN}} (\chi \lrcorner \mathcal{A}) \,. \tag{5.49}$$

Note that A does not acquire poles as was the case in subsection 5.1.2. It inherits its boundary conditions from those on $\mathcal{A}$.

We now make explicit the connection between this theory and the Bogomolny equations on $\mathbb{R}^3$. It will be convenient to introduce the 1-forms

$$\bar{e}^0 = \frac{\langle \mathrm{d}\hat{\pi}\, \hat{\pi} \rangle}{\|\pi\|^4} \,, \qquad \bar{e}^\gamma = \frac{\mathrm{d}y^{A'B'} \hat{\pi}_{A'} \hat{\pi}_{B'}}{\|\pi\|^4} \,, \qquad e^t = \mathrm{d}y^{A'B'} n_{A'B'} = \frac{\sqrt{2} i\, \mathrm{d}y^{A'B'} \pi_{A'} \hat{\pi}_{B'}}{\|\pi\|^2} \,. \tag{5.50}$$

Together with e^0, e^γ these form a frame for $T^*\mathbb{PN}$. The dual basis of vector fields includes

$$\bar\partial_0 = \|\pi\|^2\pi^{A'}\frac{\partial}{\partial\hat\pi^{A'}}\,, \qquad \bar\partial_\gamma = \pi^{A'}\pi^{B'}\partial_{A'B'}\,, \qquad \partial_t = n^{A'B'}\partial_{A'B'} = \frac{\sqrt{2}i\pi^{A'}\hat\pi^{B'}\partial_{A'B'}}{\|\pi\|^2}\,, \tag{5.51}$$

together with the obvious complex conjugates. In terms of this basis

$$\mathrm{d}' = \bar\partial_{\mathbb{MT}} + \mathrm{d}_{\mathbb{R}} = \bar e^0\bar\partial_0 + \bar e^\gamma\bar\partial_\gamma + e^t\partial_t\,. \tag{5.52}$$

We can replace d in the action (5.48) with d′, and can also remove the ambiguity in A by fixing

$$A = \bar e^0\bar A_0 + \bar e^\gamma\bar A_\gamma + e^t A_t\,. \tag{5.53}$$

Acting on our basis of 1-forms we have

$$\mathrm{d}'\bar e^0 = 0\,, \qquad \mathrm{d}'\bar e^\gamma = \frac{2\bar e^0\wedge \mathrm{d}y^{A'B'}\pi_{A'}\hat\pi_{B'}}{\|\pi\|^2} = -\sqrt{2}i\bar e^0\wedge e^t\,, \qquad \mathrm{d}'e^t = 0\,. \tag{5.54}$$

We now proceed in the usual way by trivialising $\bar A_0$ using a frame field $\hat\sigma : \mathbb{PN}\to G$. Up to gauge this can be expressed in terms of a set of group and Lie algebra fields on $\mathbb{R}^3$ determined by the boundary conditions. Performing a forbidden gauge transformation by $\hat\sigma$ brings us into the gauge $\bar A_0 = 0$ in which the classical equations of motion read

$$\begin{aligned} F(A) &= \mathrm{d}'A + [A,A]\,,\\ &= \bar e^0\wedge\bar e^\gamma\bar\partial_0\bar A_\gamma + \bar e^0\wedge e^t(\bar\partial_0 A_t - \sqrt{2}i\bar A_\gamma) + \bar e^\gamma\wedge e^t(\bar\partial_\gamma A_t - \partial_t\bar A_\gamma + [\bar A_\gamma, A_t]) = 0. \end{aligned} \tag{5.55}$$

We begin by solving

$$\bar\partial_0\bar A_\gamma = 0\,, \qquad \bar\partial_0 A_t - \sqrt{2}i\bar A_\gamma = 0\,. \tag{5.56}$$

Assuming that A is free from poles, the first of these is simply the statement that $\bar A_\gamma$ is holomorphic in π. Since $\bar A_\gamma$ has holomorphic weight 2

$$\bar A_\gamma = \pi^{A'}\pi^{B'}a_{A'B'} \tag{5.57}$$

where $a_{A'B'} = a_{B'A'}$ depends only on y. Substituting this into the second equation gives

$$\bar\partial_0 A_t = \sqrt{2}i\pi^{A'}\pi^{B'}a_{A'B'}\,, \tag{5.58}$$

the solution to which is

$$A_t = \frac{\sqrt{2} i \pi^{A'} \hat{\pi}^{B'} a_{A'B'}}{\|\pi\|^2} + i\varphi = n^{A'B'} a_{A'B'} + i\varphi \,. \tag{5.59}$$

Here φ also depends only on y. In full we therefore have

$$A = \bar{e}^{\gamma} \pi^{A'} \pi^{B'} a_{A'B'} + e^t (n^{A'B'} a_{A'B'} + i\varphi) \,. \tag{5.60}$$

Recall that there is a redundancy in the choice of frame field $\hat{\sigma} \mapsto h\hat{\sigma}$ for h independent of π. Under this transformation A and φ transform as a connection and adjoint valued scalar on $\mathbb{R}^3$. The boundary conditions on A allow us to express a and φ in terms of the group and Lie algebra fields determining $\hat{\sigma}$. At this point the π dependence of A is in principle completely fixed, and integrating over the fibres of $\mathbb{PN} \to \mathbb{R}^3$ gives a 3d spacetime action.

Its classical equations of motion will imply

$$\bar{\partial}_{\gamma} A_t - \partial_t \bar{A}_{\gamma} + [\bar{A}_{\gamma}, A_t] = \frac{\sqrt{2} i \pi^{A'} \pi^{B'} \pi^{C'} \hat{\pi}^{D'}}{\|\pi\|^2} \left(f_{A'B'C'D'} + \frac{1}{\sqrt{2}} \varepsilon_{C'D'} \nabla_{A'B'} \varphi \right) = 0 \,, \tag{5.61}$$

where $\nabla = \mathrm{d} + a$ and $f = f(a)$ is the curvature. Decomposing $\pi^{A'} \pi^{B'} \pi^{C'} \hat{\pi}^{D'}$ into its totally symmetric and mixed parts

$$4\pi^{A'} \pi^{B'} \pi^{C'} \hat{\pi}^{D'} = 4\pi^{(A'} \pi^{B'} \pi^{C'} \hat{\pi}^{D')} - \|\pi\|^2 (\pi^{A'} \pi^{B'} \varepsilon^{C'D'} + \pi^{A'} \pi^{C'} \varepsilon^{B'D'} + \pi^{B'} \pi^{C'} \varepsilon^{A'D'}) \tag{5.62}$$

equation (5.61) is

$$\pi^{A'} \pi^{B'} \varepsilon^{C'D'} f_{A'C'B'D'} = \frac{1}{\sqrt{2}} \pi^{A'} \pi^{B'} \nabla_{A'B'} \varphi \,. \tag{5.63}$$

We then identify

$$(*f)_{A'B'} = \sqrt{2} f_{A'C'B'D} \varepsilon^{C'D'} \,, \tag{5.64}$$

and so the Bogomolny equation follows

$$* f = \nabla \varphi \,. \tag{5.65}$$

This is essentially a consequence of the Hitchin-Ward correspondence.

So far we have not been specific about our choice of Φ and associated boundary conditions, except in assuming that we are not permitting any poles in A. By choosing

Φ as in chapter 4 we obtain a range of spacetime actions which are all straightforward to compute as symmetry reductions of those appearing therein. If Φ is nowhere vanishing then we need not allow poles in A, and so the classical equations of motion of the resulting 3d theory will be equivalent to the Bogomolny equations. It is also straightforward to generalise the reality conditions in section 4.5 to the 5d case, though they remain rather stringent.

Remark. We could also perform an analogous reduction by a non-null translation in ultrahyperbolic signature. This leads to CS_5 on the projective spinor bundle of 3d Minkowski space $\mathbb{R}^{2,1}$. Under this reduction the partial connection acquires a simple pole along the locus $\pi \sim \bar{\pi}$.

In Lorentzian signature the reality conditions on Φ are less stringent. $\Phi = 1/\langle\pi\,\alpha\rangle^2\langle\pi\beta\rangle^2$ leads to the $2+1$ dimensional chiral model [163] if α, β are real, and [117] if they are conjugate. $\Phi = 1/\langle\pi\,\alpha\rangle^4$ for α real leads to the pseudo-dual of the 2+1 dimensional chiral model [63].

5.3 Discussion

In this chapter we have seen that CS_4 naturally arises as a symmetry reduction of CS_6 on twistor correspondence space. This connects two different approaches to the study of integrable systems: as symmetry reductions of the ASDYM equations, and as defects in CS_4. In particular we have obtained the CS_4 realisations of the PCM in both standard and trigonometric forms, the pseudo-dual of the PCM and the integrable coupled σ-models of [58, 59]. Notably the pseudo-dual of the PCM had not previously been obtained from CS_4, and the integrable coupled σ-models are not known to arise as a symmetry reduction of the ASDYM equations.

We also performed the symmetry reduction of HCS on twistor space by a 1 dimensional group of translations, obtaining a CS_5 theory on minitwistor correspondence space describing the Bogomolony equations.

So far we have performed only the simplest possible reductions by groups of translations with orbits on which the induced metric is non-degenerate. It would clearly be worthwhile to consider more general reductions. The non-linear Schrödinger and KdV equations arise as symmetry reductions of the ASDYM equations on $\mathbb{R}^{2,2}$ by 2d groups of translations with orbits on which the metric has rank 1. It would be interesting to obtain CS_4 descriptions of both. (KdV can already be obtained from CS_4, see [74, 78]. It seems that symmetry reduction would lead to a rather different realisation, however.) Ordinary and extended Toda theory arise as symmetry reductions of the ASDYM equations by 2d groups of translations together with rotations. We expect that both can be realised in CS_4 since they have descriptions as affine Gaudin models [159]. (Note also the rather different proposal in [8] relating CS_4 to a 3d Toda theory.) Finally the Ernst equation arises as a symmetry reduction of the ASDYM equations by a translation and rotation [171, 161]. In [140] this was obtained starting with HCS on $\mathbb{PT}$, but its CS_4 counterpart was not determined.

Another proposal is to consider more general choices of Ω and corresponding boundary conditions. Some steps in this direction have already been taken in [36]. Note also that in [53] integrable coset models were obtained by orbifolding the CS_4 realisation of integrable coupled σ-models. We expect that orbifolding in the same way on $\mathbb{PT}$ will lead to 4d integrable coset models which under symmetry reduction descend to their 2d counterparts.

It would be intriguing to see whether any of this can be applied in the quantum case. The first step towards doing so would be to find an appropriate counterpart of symmetry reduction. One proposal is to compactify in the directions one wishes to reduce and then take the small volume limit. This is likely related to recent work connecting CS_4 and HCS using topological string dualities [173]. Alternatively, we could try to supersymmetrise in the directions we wish to quotient, and then topologically twist.

Finally, it may also be worthwhile studying some of the symmetry reductions as quantum theories in their own right. For example, it seems likely that CS_5 introduced in section 5.2 might evade some of the anomalies suffered by HCS [48].

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Appendix A

Computations

A.1 Evaluation of equations (2.100) and (2.109)

In this appendix we evaluate the necessary integrals involved in computing the OPE of a bulk and boundary Wilson line in section 2.3.1.

Let's start by determining the constant λ_+ by evaluating the integral in equation (2.100). It is equal to

$$\lambda_+ = \int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbb{C}} \mathrm{d}^2 z \, z I_+(x, z, \bar{z}, 1) \;, \tag{A.1}$$

where

$$I_+(x, z, \bar{z}, 1) = \frac{\bar{z}}{(x^2 + |z|^2)^{3/2}} \left(\frac{1}{((x-1)^2 + |z|^2)^{3/2}} + \frac{1}{((x+1)^2 + |z|^2)^{3/2}} \right) . \tag{A.2}$$

We have

$$\begin{aligned} \lambda_+ &= 2 \int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbb{C}} \mathrm{d}^2 z \, \frac{|z|^2}{(x^2 + |z|^2)^{3/2}((x-1)^2 + |z|^2)^{3/2}} \\ &= -8\pi i \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}r \, \frac{r^3}{(x^2 + r^2)^{3/2}((x-1)^2 + r^2)^{3/2}} \, . \end{aligned} \tag{A.3}$$

where in the first equality we've mapped $x \mapsto -x$ in the second term of I_+, and in going to the second line we've performed the integral over the phase of z. Using the Feynman parametrization

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}t \, \frac{t^{\alpha-1}(1-t)^{\beta-1}}{(tA + (1-t)B)^{\alpha+\beta}} \, . \tag{A.4}$$

we can rewrite this as

$$\lambda_+ = -64i \int_0^1 \mathrm{d}t\, t^{1/2}(1-t)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}r\, \frac{r^3}{(r^2 + t(x-1)^2 + (1-t)x^2)^3} \,. \tag{A.5}$$

The integral over r can be performed directly since, using the substitution $r = a\tan\phi$,

$$\int_0^{\infty} \mathrm{d}r\, \frac{r^3}{(r^2+a^2)^3} = \frac{1}{a^2}\int_0^{\pi/2} \mathrm{d}\varphi\, \cos\varphi\, \sin^3\varphi = \frac{1}{4a^2} \,. \tag{A.6}$$

This gives

$$\lambda_+ = -16i \int_0^1 \mathrm{d}t\, t^{1/2}(1-t)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{t(x-1)^2 + (1-t)x^2} \,. \tag{A.7}$$

Noticing that $t(x-1)^2 + (1-t)x^2 = x^2 - 2xt + t = (x-t)^2 + t(1-t)$, the integral over x is elementary and we obtain

$$\lambda_+ = -16\pi i \int_0^1 \mathrm{d}t\, \frac{t^{1/2}(1-t)^{1/2}}{t^{1/2}(1-t)^{1/2}} = -16\pi i \,, \tag{A.8}$$

which is the value used in the main text.

Next we determine the constant μ_+ by evaluating the integral in equation (2.109). We have

$$\mu_+ = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}x \int_{\mathbb{C}} \mathrm{d}^2 z\, \frac{|z|^2}{((x-1)^2 + |z|^2)^{3/2}((x+1)^2 + |z|^2)^{3/2}} \,. \tag{A.9}$$

Making the substitutions $x = 2\tilde{x} - 1$ and $z = 2\tilde{z}$, this integral can be related to λ_+.

$$\mu_+ = \frac{1}{8}\int_{-\infty}^{\infty} \mathrm{d}\tilde{x} \int_{\mathbb{C}} \mathrm{d}^2 \tilde{z}\, \frac{|\tilde{z}|^2}{((\tilde{x}-1)^2 + |\tilde{z}|^2)^{3/2}(\tilde{x}^2 + |\tilde{z}|^2)^{3/2}} = \frac{\lambda_+}{16} \,. \tag{A.10}$$

We conclude that

$$\mu_+ = -\pi i \tag{A.11}$$

which is used in the text.

A.2 Classification of constant lifts in the trigonometric case

In this appendix we classify involutive automorphisms of the Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \widetilde{\mathfrak{h}}$ which swap the Lagrangian subalgebras $\mathfrak{g}_\pm$ and preserve the invariant bilinear. These automorphisms are used in section 3.2 to generate trigonometric K-matrices.

Recall that $\mathfrak{g}_0$ is a complex simple Lie algebra, and $\widetilde{\mathfrak{h}}$ is a second copy of its Cartan subalgebra. The subalgebras $\mathfrak{g}_\pm$ are defined by

$$\mathfrak{g}_+ = \mathfrak{n}_+ \dotplus \{(H, i\widetilde{H}) | H \in \mathfrak{h}\} , \qquad \mathfrak{g}_- = \mathfrak{n}_- \dotplus \{(H, iM(\widetilde{H})) | H \in \mathfrak{h}\} . \tag{A.12}$$

for $M \in \operatorname{End} \widetilde{\mathfrak{h}}$. Recall that for these subalgebras to be disjoint M must not have a $+1$ eigenvalue. We seek involutive automorphisms σ of $\mathfrak{g}$ such that

$$\sigma(\mathfrak{g}_+) = \mathfrak{g}_- . \tag{A.13}$$

We also require that they preserve the invariant bilinear

$$\operatorname{tr}_{\mathfrak{g}} = \operatorname{tr}_{\mathfrak{g}_0} + \operatorname{tr}_{\widetilde{\mathfrak{h}}} , \tag{A.14}$$

where $\operatorname{tr}_{\mathfrak{g}_0}$ is the minimal $\mathfrak{g}_0$-invariant bilinear on $\mathfrak{g}_0$, and $\operatorname{tr}_{\widetilde{\mathfrak{h}}} = \operatorname{tr}_{\mathfrak{g}_0}|_{\widetilde{\mathfrak{h}}}$ is its restriction to the Cartan. For $\mathfrak{g}_-$ to be isotropic M must be orthogonal with respect to $\operatorname{tr}_{\widetilde{\mathfrak{h}}}$.

We begin by decomposing our automorphism with respect to the direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \widetilde{\mathfrak{h}}$, writing

$$\sigma = \begin{pmatrix} \chi & \phi \\ \psi & \omega \end{pmatrix} . \tag{A.15}$$

The centre of $\mathfrak{g}$ is $\widetilde{\mathfrak{h}}$, and any Lie algebra automorphism preserves the centre, so we must have $\sigma(\widetilde{\mathfrak{h}}) = \widetilde{\mathfrak{h}}$. Hence $\phi = 0$. Since $\operatorname{tr}_{\mathfrak{g}}(\widetilde{H}X) = 0$ for any $X \in \mathfrak{g}_0$ and $\widetilde{H} \in \widetilde{\mathfrak{h}}$, if σ preserves $\operatorname{tr}_{\mathfrak{g}}$, then

$$0 = \operatorname{tr}_{\mathfrak{g}}(\sigma(\widetilde{H})\sigma(X)) = \operatorname{tr}_{\mathfrak{g}}(\omega(\widetilde{H}), \chi(X) + \psi(X)) = \operatorname{tr}_{\widetilde{\mathfrak{h}}}(\omega(\widetilde{H}), \psi(X)) . \tag{A.16}$$

The bilinear $\operatorname{tr}_{\widetilde{\mathfrak{h}}}$ is non-degenerate, so $\psi = 0$.

σ being involutive also implies that $\chi^2 = \mathbb{1}_{\mathfrak{g}_0}$, so that χ is an involutive automorphism of $\mathfrak{g}_0$. For σ to swap the subalgebras $\mathfrak{g}_\pm$, χ must exchange the the Borel subalgebras

$$\mathfrak{b}_+ = \mathfrak{n}_+ \dotplus \mathfrak{h} , \qquad \mathfrak{b}_- = \mathfrak{n}_- \dotplus \mathfrak{h} . \tag{A.17}$$

Since $\chi(\mathfrak{b}_\pm) = \mathfrak{b}_\mp$

$$\chi(\mathfrak{h}) \subset (\mathfrak{b}_+ \cap \mathfrak{b}_-) = \mathfrak{h}\,. \tag{A.18}$$

An automorphism preserving the Cartan must map root spaces to other root spaces. Furthermore, since χ swaps the Borel subalgebras $\mathfrak{b}_\pm$ it must swap positive and negative roots. This restricts the action of χ on the roots to be the composition of multiplication by -1 with an involutive automorphism of the Dynkin diagram, which we will denote by $\gamma \in \operatorname{Sym}\Delta$.

Now let s and π_γ be the involutive automorphisms

$$\begin{aligned} s : h_\mu &\mapsto -h_\mu\,, \qquad e_\mu \mapsto f_\mu\,, \qquad f_\mu \mapsto e_\mu \\ \pi_\gamma : h_\mu &\mapsto h_{\gamma(\mu)}\,, \qquad e_\mu \mapsto e_{\gamma(\mu)}\,, \qquad f_\mu \mapsto f_{\gamma(\mu)} \end{aligned} \tag{A.19}$$

in terms of the Chevalley basis $\{f_\mu, e_\mu\}_{\mu\in\Phi_+} \cup \{h_\mu\}_{\mu\in\Delta}$ of $\mathfrak{g}_0$. (The fact that the above can be uniquely extended to involutive automorphisms of $\mathfrak{g}_0$ is a standard result in elementary Lie algebra theory. See, e.g., [91].) When $\mu \in \Delta$, we find that the composition $\pi_\gamma \circ s \circ \chi$ fixes $\mathfrak{h}$ pointwise. Any automorphism fixing $\mathfrak{h}$ pointwise must fix the root spaces, and so the only freedom remaining in $\pi_\gamma \circ s \circ \chi$ is to map $e_\mu \mapsto \Lambda_\mu e_\mu$ for $\mu \in \Delta$, where $\Lambda_\mu \in \mathbb{C}^*$. Since $[e_\mu, f_\mu] = h_\mu$ we must then have $f_\mu \mapsto \Lambda_\mu^{-1} f_\mu$ also. In summary

$$\chi = s \circ \pi_\gamma \circ \exp(\mathrm{ad}_\lambda)\,, \tag{A.20}$$

where $\lambda = \lambda^\mu h_\mu \in \mathfrak{h}$ is given by $\exp(\sum_\nu A_{\mu\nu}\lambda^\nu) = \Lambda_\mu$ for A the Cartan matrix of $\mathfrak{g}_0$. For this χ to be involutive we require that λ lies in the $+1$ eigenspace of $\pi_\gamma|_{\mathfrak{h}}$.

Whether we get a solution of the 'soliton preserving' or 'soliton reversing' BYBE depends on whether χ is an inner or outer automorphism. If $\mathfrak{g}_0$ is not of type $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_{2n}(\mathbb{C})$ or $\mathfrak{e}_6$ then there are no outer automorphisms, and so χ is necessarily inner. In the remaining cases, certainly $\exp(\mathrm{ad}_\lambda)$ is inner, any non-trivial π_γ is outer, and s can be inner or outer depending on $\mathfrak{g}_0$. In fact, s is outer for $\mathfrak{g}_0 = \mathfrak{sl}_n(\mathbb{C})$, inner for $\mathfrak{g}_0 = \mathfrak{so}_{2n}(\mathbb{C})$ with n even, outer for $\mathfrak{g}_0 = \mathfrak{so}_{2n}(\mathbb{C})$ with n odd, and outer for $\mathfrak{g}_0 = \mathfrak{e}_6$. (Here $n \geq 3$.)

Having fixed χ it remains to determine ω. For σ to preserve $\mathrm{tr}_{\mathfrak{g}}$ we must have $\omega \in O(\widetilde{\mathfrak{h}})$, while for it to be involutive we must have $\omega^2 = \mathbb{1}_{\widetilde{\mathfrak{h}}}$. We have yet to impose $\sigma(\mathfrak{g}_+) = \mathfrak{g}_-$. Consider $(H, i\widetilde{H}) \in \mathfrak{g}_+$ for some $H \in \mathfrak{h}$. Acting with σ we learn that

$$\sigma((H, i\widetilde{H})) = (\chi(H), i\omega(\widetilde{H})) = (H', iM(\widetilde{H}')) \in \mathfrak{g}_- \tag{A.21}$$

for some $H' = \chi(H) \in \mathfrak{h}$, so we must have $\omega = M \circ \chi|_{\widetilde{\mathfrak{h}}}$. (Here we've restricted χ to the Cartan and interpreted it as an automorphism of $\widetilde{\mathfrak{h}}$.) Since we've already imposed the condition that σ be involutive, this is sufficient to ensure that $\sigma(\mathfrak{g}_-) = \mathfrak{g}_+$. The condition $\omega^2 = \mathbb{1}_{\widetilde{\mathfrak{h}}}$ imposes the constraint

$$(M \circ \chi|_{\widetilde{\mathfrak{h}}})^2 = \mathbb{1}_{\widetilde{\mathfrak{h}}} \tag{A.22}$$

on M. Since $M, \chi|_{\widetilde{\mathfrak{h}}} \in O(\widetilde{\mathfrak{h}})$, it follows immediately that $\omega = M \circ \chi \in O(\widetilde{\mathfrak{h}})$.

In the case that we choose γ to be the identity, we have $\chi|_{\widetilde{\mathfrak{h}}} = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$, and M obeys $M^2 = \mathbb{1}_{\widetilde{\mathfrak{h}}}$. Any involutive M is diagonalizable with $+1$ and -1 eigenspaces, however M cannot have a $+1$ eigenvalue. We deduce that $M = -\mathbb{1}_{\widetilde{\mathfrak{h}}}$, and $\omega = \mathbb{1}_{\widetilde{\mathfrak{h}}}$. On the other hand, if we choose γ to be a non-trivial automorphism of the Dynkin diagram then there exist non-trivial choices for M.

A.3 Involutive automorphisms of loop algebras

In subsection 3.2.2 we argue that trigonometric K-matrices with z-dependent classical limits can be obtained using our orbifold construction. This requires lifting the $\mathbb{Z}_2$ action to the adjoint bundle of CS_4 using certain z-dependent automorphisms of the Manin triple $\mathfrak{g} = \mathfrak{g}_0 + \widetilde{\mathfrak{h}}$. In this appendix we show how suitable automorphisms can be obtained by studying the loop algebra of $\mathfrak{g}$.

This is defined to be the Lie algebra of finite Laurent series in the formal parameter u, and is denoted by $L\mathfrak{g} = \mathfrak{g}[u, u^{-1}]$. It inherits the structure of a Manin triple from $\mathfrak{g}$. In particular, it admits the invariant bilinear

$$(a(u), b(u))_{L\mathfrak{g}} = \oint \frac{\mathrm{d}u}{u} \operatorname{tr}_{\mathfrak{g}}(a(u)b(u)) \,, \tag{A.23}$$

and Lagrangian subalgebras

$$(L\mathfrak{g})_- = \mathfrak{g}[u^{-1}] \dotplus \mathfrak{g}_- \qquad \text{and} \qquad (L\mathfrak{g})_+ = \mathfrak{g}[u] \dotplus \mathfrak{g}_+ \,. \tag{A.24}$$

We seek automorphisms $\hat{\sigma} : L\mathfrak{g} \to L\mathfrak{g}$ which can be interpreted as u-dependent automorphisms $\hat{\sigma} : \mathbb{C}^* \to \operatorname{Aut}\mathfrak{g}$ obeying $\hat{\sigma}(u^{-1})\hat{\sigma}(u) = \mathbb{1}_{\mathfrak{g}}$. The loop algebra admits an involution $\iota : u \mapsto u^{-1}$, and this latter condition is equivalent to $\theta = \hat{\sigma} \circ \iota$ being involutive. We also require that θ preserves the bilinear $(\cdot, \cdot)_{L\mathfrak{g}}$ and exchanges the subalgebras $(L\mathfrak{g})_{\pm}$. (This ensures that it leads to a $\hat{\sigma}(u)$ which will swap the boundary conditions at $u = 0, \infty$ in CS_4. See subsection 3.2.2 for details.)

To find such automorphisms, we first quotient by the centre, $L\widetilde{\mathfrak{h}}$. θ descends to the quotient, and determines an involutive automorphism of $L\mathfrak{g}_0$ swapping the subalgebras

$$\mathfrak{b}_- = \mathfrak{h} \dotplus \mathfrak{n}_- \dotplus \mathfrak{g}_0[u^{-1}]\,, \qquad \mathfrak{b}_+ = \mathfrak{h} \dotplus \mathfrak{n}_+ \dotplus \mathfrak{g}_0[u]\,. \tag{A.25}$$

It can then be lifted to the affine algebra $\widehat{\mathfrak{g}}_0'$, where it exchanges two Borel subalgebras. (These are the $\mathfrak{b}_\pm$ defined above with $\mathfrak{h}$ extended to include the central element c.) Such automorphisms can easily be classified following arguments presented in [97] and the appendix of [99]. One finds that all automorphisms swapping these subalgebras are of the form

$$\theta = \mathrm{Ad}\,\Lambda \circ \Gamma \circ \omega\,, \tag{A.26}$$

where

$$\omega : (e_\mu, f_\mu, h_\mu) \mapsto (f_\mu, e_\mu, -h_\mu) \qquad \text{for } \mu \in \widehat{\Delta} \tag{A.27}$$

is the Chevalley involution, Γ extends the action of a permutation of the Dynkin diagram of $\widehat{\mathfrak{g}}_0'$, γ, to all of $\widehat{\mathfrak{g}}_0'$ by

$$\Gamma : (e_\mu, f_\mu, h_\mu) \mapsto (e_{\gamma(\mu)}, f_{\gamma(\mu)}, h_{\gamma(\mu)}) \qquad \text{for } \mu \in \widehat{\Delta}\,, \tag{A.28}$$

and finally

$$\mathrm{Ad}\,\Lambda : (e_\mu, f_\mu, h_\mu) \mapsto (\Lambda(\alpha_\mu) e_\mu, \Lambda(\alpha_\mu)^{-1} f_\mu, h_\mu) \tag{A.29}$$

for some γ-invariant map $\Lambda : \widehat{\Delta} \mapsto \mathbb{C}^*$. Here $\widehat{\Delta}$ denotes the set of simple roots of the affine algebra $\widehat{\mathfrak{g}}_0'$, and $\{e_\mu, f_\mu, h_\mu\}$ are its Chevalley generators. Given such a θ we can restrict it to the loop algebra $L\mathfrak{g}_0$, and then extend it to $L\widetilde{\mathfrak{h}}$ by defining $\theta|_{\widetilde{\mathfrak{h}}} = M \circ \theta|_{\mathfrak{h}}$. Indeed this is the only way of extending it which is consistent with the fact that θ must swap $(L\mathfrak{g})_\pm$. Finally by taking the composition $\theta \circ \iota$ we hope to recover a z-dependent automorphisms of $\mathfrak{g}$. Unfortunately this is not always the case. (A simple example of an automorphisms of $L\mathfrak{g}$ which is not a z-dependent automorphism of $\mathfrak{g}$ is the map $z \mapsto \lambda z$ for $z \in \mathbb{C}$.) Fortunately by judiciously choosing Λ we can generate automorphisms of the desired form.

This is how the examples of z-dependent automorphisms in section 3.2.3 were obtained.

A.4 Classification of lifts in the elliptic case

In this appendix we classify all bundle maps $\hat{\sigma} : \operatorname{Ad} P \to \operatorname{Ad} P$ covering $z \mapsto -z$, i.e., which make the following diagram commute.

$$\begin{array}{ccc} \operatorname{Ad} P & \xrightarrow{\hat{\sigma}} & \operatorname{Ad} P \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathrm{T}^2_\tau & \xrightarrow{z \mapsto -z} & \mathrm{T}^2_\tau \end{array} \tag{A.30}$$

These are used in section 3.3 to generate elliptic solutions to the BYBE.

Proof. Recall from section 1.3.5 that the total space of the vector bundle $\operatorname{Ad} P \to \mathrm{T}^2_\tau$ is the quotient of $\mathbb{C} \times \mathfrak{sl}_n(\mathbb{C})$ by the equivalence relation $(z, X) \sim (z + a + b\tau, \operatorname{conj}(A^{-\zeta a} B^{-b})(X))$ for $a, b \in \mathbb{Z}$. We can lift $\hat{\sigma}$ by the quotient map to

$$\hat{\sigma} : \mathbb{C} \times \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C} \times \mathfrak{sl}_n(\mathbb{C}) \,, \tag{A.31}$$

where explicitly

$$\hat{\sigma} : (z, X) \mapsto (-z, \hat{\sigma}(z)(X)) \,. \tag{A.32}$$

Here we abuse notation by interpreting $\hat{\sigma} : \mathbb{C} \to \operatorname{End}(\mathfrak{sl}_n(\mathbb{C}))$. Since $\hat{\sigma}$ preserves the Lie algebra structure on the fibres of $\operatorname{Ad} P$, its image is contained in $\operatorname{Aut}(\mathfrak{sl}_n(\mathbb{C}))$. In order to preserve the vacuum, $\hat{\sigma}$ must be holomorphic in $\mathbb{C}$, and to descend to $\operatorname{Ad} P$ it must have quasi-periodicity

$$\hat{\sigma}(z + a + b\tau) = \operatorname{conj}(A^{\zeta a} B^b) \circ \hat{\sigma}(z) \circ \operatorname{conj}(A^{\zeta a} B^b) \tag{A.33}$$

for all $z \in \mathbb{C}$ and $a, b \in \mathbb{Z}$.

Quasi-periodicity ensures that $\hat{\sigma}$ is bounded, and so by Liouville's theorem must be constant. We write $\hat{\sigma}(z) = \sigma$. Letting both sides of (A.33) act on the generator $t_{i,j}$ we learn that

$$\sigma(t_{i,j}) = \epsilon^{ai+bj} \operatorname{conj}(A^{\zeta a} B^b)(\sigma(t_{i,j})) \tag{A.34}$$

for all $a, b \in \mathbb{Z}$. Hence $\sigma(t_{i,j})$ is an eigenvector of $\operatorname{conj} A^\zeta$ & $\operatorname{conj} B$ with eigenvalues ϵ^{-i} & ϵ^{-j} respectively. These eigenspaces are 1 dimensional, hence

$$\sigma(t_{i,j}) = \mu_{i,j} t_{-i,-j} \tag{A.35}$$

for $\mu_{i,j} \in \mathbb{C}$. The $\mu_{i,j}$ must be non-vanishing to ensure that σ is bijective.

We have yet to impose that σ is an automorphism of $\mathfrak{sl}_n(\mathbb{C})$. In the basis $\{t_{i,j}\}_{(i,j)\in\mathcal{I}_n}$ the Lie bracket is given by

$$[t_{i,j}, t_{k,\ell}] = (\epsilon^{-\zeta^{-1}jk} - \epsilon^{-\zeta^{-1}i\ell})t_{i+k,j+\ell}\,, \tag{A.36}$$

so we learn that

$$(\mu_{i+k,j+\ell} - \mu_{i,j}\mu_{k,\ell})(\epsilon^{-\zeta^{-1}jk} - \epsilon^{-\zeta^{-1}i\ell}) = 0\,. \tag{A.37}$$

Thus $\mu_{i+k,j+\ell} = \mu_{i,j}\mu_{k,\ell}$ whenever $jk \not\equiv i\ell\,(n)$. This allows to build up $\mu_{i,j}$ recursively from $\mu_{1,0}$ & $\mu_{0,1}$, giving $\mu_{i,j} = \mu_{1,0}^i\mu_{0,1}^j$ for all $(i,j)\in\mathcal{I}_n$.

Finally, notice that $\mu_{0,1}^{n+1}\mu_{1,0} = \mu_{0,1}\mu_{1,0}$ so that $\mu_{1,0}^n = 1$. This implies that $\mu_{1,0} = \epsilon^\xi$ for some $\xi\in\mathbb{Z}_n$, and similarly $\mu_{0,1} = \epsilon^\eta$ for some $\eta\in\mathbb{Z}_n$. We conclude that

$$\hat{\sigma} : (z, t_{i,j}) \mapsto (-z, \sigma(t_{i,j})) = (-z, \epsilon^{i\xi+j\eta}t_{-i,-j})\,. \tag{A.38}$$

These are the lifts used in section 3.3.

Finally note that σ is involutive, and also obeys

$$\sigma\circ\mathrm{conj}_{A^\zeta} = \mathrm{conj}_{A^{-\zeta}}\circ\sigma\,, \qquad \sigma\circ\mathrm{conj}_B = \mathrm{conj}_{B^{-1}}\circ\sigma. \tag{A.39}$$

This shows that σ reverses the holonomies around the cycles of T^2_τ, as expected. □

A.5 Matching elliptic ℓ- and K-matrices

In this appendix we show that the elliptic K-matrices appearing in [100] agree to first order in $\hbar$ with a subset of the classical ℓ-matrices obtained from CS_4 in subsection 3.3.1. For a brief explanation of how we perform this comparison see appendix B.2.

The K-matrices in [100] depend on four complex parameters λ_*, indexed by the pair $r_*, s_*\in\{0,1\}$, together with one further complex parameter Λ. This gives five complex degrees of freedom in total, one of which describes the overall scale of the K-matrix. In terms of the standard basis of $\mathrm{End}\,(\mathbb{C}^n)$, $\{E_\alpha{}^\beta\}_{\alpha,\beta\in\mathbb{Z}_n}$, they take the form

$$K(z) = K(z;\lambda_*,\Lambda) = \sum_{r_*,s_*\in\{0,1\}}\lambda_*\sum_{\alpha,\beta\in\mathbb{Z}_n}K^\alpha{}_\beta(z;r_*,s_*,\Lambda)E_\alpha{}^\beta\,, \tag{A.40}$$

for

$$K^{\alpha}{}_{\beta}(z;r_*,s_*,\Lambda) = \frac{\theta\left[\begin{smallmatrix}1/2-2\beta/n\\1/2\end{smallmatrix}\right](2\Lambda-2z|n\tau)\theta\left[\begin{smallmatrix}s_*/2-\alpha/n\\nr_*/2\end{smallmatrix}\right](\Lambda+z|n\tau)\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](2\Lambda|n\tau)}{\theta\left[\begin{smallmatrix}1/2+(\alpha-\beta)/n\\1/2\end{smallmatrix}\right](-2z|n\tau)\theta\left[\begin{smallmatrix}s_*/2-\beta/n\\nr_*/2\end{smallmatrix}\right](\Lambda-z|n\tau)\theta\left[\begin{smallmatrix}1/2-(\alpha+\beta)/n\\1/2\end{smallmatrix}\right](\Lambda|n\tau)}\,. \tag{A.41}$$

Following the procedure outlined in appendix B.2, we first need to identify a suitable classical limit of the above. Setting $\Lambda = 0$ gives

$$K(z;\lambda_*,0) = \sum_{r_*,s_*\in\{0,1\}}\lambda_*\sum_{\alpha,\beta\in\mathbb{Z}_n}(-)^{nr_*s_*}\delta_{\alpha+\beta\equiv 0\,(n)}E_{\alpha}{}^{\beta} = \sum_{r_*,s_*\in\{0,1\}}\lambda_*(-)^{nr_*s_*}\sum_{\alpha\in\mathbb{Z}_n}E_{\alpha}{}^{-\alpha}\,. \tag{A.42}$$

We recognise $\sum_{\alpha\in\mathbb{Z}_n}E_{\alpha}{}^{-\alpha}$ as the image of $\tau\in\mathrm{SL}_n(\mathbb{C})$ from equation (3.67) with $\xi=\eta=0$ in the fundamental representation. It satisfies

$$\mathrm{conj}(\tau_U)(T_{i,j}) = T_{-i,-j}\,. \tag{A.43}$$

τ_U is therefore a sensible classical limit for the K-matrix, and it natural to normalise by imposing

$$\sum_{r_*,s_*\in\{0,1\}}\lambda_*(-)^{nr_*s_*} = 1\,. \tag{A.44}$$

The next step is to replace the parameters $\lambda_*,\Lambda\in\mathbb{C}$ with formal power series $\hat{\lambda}_*,\hat{\Lambda}\in\mathbb{C}[[\hbar]]$ obeying

$$\hat{\lambda}_* = \lambda_* + \mathcal{O}(\hbar)\,,\qquad \hat{\Lambda} = \hbar\rho + \mathcal{O}(\hbar^2)\,. \tag{A.45}$$

This ensures that the classical limit of (A.40) is τ_U.

Now that we've introduced $\hbar$ into the K-matrix we can identify the corresponding classical ℓ-matrix. We proceed by decomposing $L(z) = L(z;\hat{\lambda}_*,\hat{\Lambda}) = K(z;\hat{\lambda}_*,\hat{\Lambda})\tau_U^{-1}$ in the basis $\{T_{i,j}\}_{i,j\in\mathbb{Z}_n}$.

$$L(z) = L(z;\hat{\lambda}_*,\hat{\Lambda}) = \sum_{r_*,s_*\in\{0,1\}}\hat{\lambda}_*\sum_{i,j\in\mathbb{Z}_n}\widetilde{L}_*^{i,j}(z;\hat{\Lambda})T_{i,j} \tag{A.46}$$

for

$$\widetilde{L}_*^{i,j}(z;\hat{\Lambda}) \\ = \frac{\theta\left[{}^{1/2}_{1/2}\right](2\hat{\Lambda}|n\tau)}{n\theta\left[{}^{1/2-i/n}_{1/2}\right](2\hat{\Lambda}|n\tau)} \sum_{k\in\mathbb{Z}_n} \epsilon^{jk} \frac{\theta\left[{}^{1/2+2k/n}_{1/2}\right](2\hat{\Lambda}-2z|n\tau)\theta\left[{}^{s_*/2-(i+k)/n}_{nr_*/2}\right](\hat{\Lambda}+z|n\tau)}{\theta\left[{}^{1/2+(i+2k)/n}_{1/2}\right](-2z|n\tau)\theta\left[{}^{s_*/2+k/n}_{nr_*/2}\right](\hat{\Lambda}-z|n\tau)}\,. \tag{A.47}$$

Here the subscript $*$ on $\widetilde{L}_*^{i,j}(z;\hat{\Lambda})$ indicates dependence on r_*, s_*. We can characterise $\widetilde{L}_*^{i,j}(z;\hat{\Lambda})$ by its quasi-periodicities

$$\widetilde{L}_*^{i,j}(z+1;\hat{\Lambda}) = \epsilon^i \widetilde{L}_*^{i,j}(z;\hat{\Lambda})\,, \qquad \widetilde{L}_*^{i,j}(z+\tau;\hat{\Lambda}) = \epsilon^j e^{2\hat{\Lambda}} \widetilde{L}_*^{i,j}(z;\hat{\Lambda})\,, \tag{A.48}$$

and pole structure in the fundamental domain, within which $L^{i,j}$ has simple poles at $z = 0$, $1/2$, $\tau/2$ and $(1+\tau)/2$. The residues differ qualitatively for n even and odd.

For n odd

$$\mathrm{Res}_{z=(a_*+b_*\tau)/2}\, \widetilde{L}_*^{i,j}(z;\hat{\Lambda}) \\ = -\frac{1}{2n}\varepsilon^{-2^{-1}(i-b^*)(j-a^*)}(-)^{a_*s_*+b_*r_*} e^{\pi i b^*(2\hat{\Lambda}+a^*)/n} \Delta(\hat{\Lambda}|n\tau)\,, \tag{A.49}$$

where $a_*, b_* \in \{0,1\}$ and we have defined

$$\Delta(z|\tau) = \frac{\theta\left[{}^{1/2}_{1/2}\right](z|\tau)}{\theta\left[{}^{1/2}_{1/2}\right]'(0|\tau)}\,. \tag{A.50}$$

Rescaling by

$$-\frac{n\theta\left[{}^{1/2}_{1/2}\right]'(0|n\tau)\theta\left[{}^{1/2}_{1/2}\right](-2\hat{\Lambda}/n|\tau)}{\theta\left[{}^{1/2}_{1/2}\right]'(0|\tau)\theta\left[{}^{1/2}_{1/2}\right](2\hat{\Lambda}|n\tau)} = -\frac{n\Delta(-2\hat{\Lambda}/n,\tau)}{\Delta(2\hat{\Lambda},n\tau)} = 1 + \mathcal{O}(\hbar)\,, \tag{A.51}$$

and introducing the parameters $\hat{\mu}_*$

$$\hat{\mu}_* = \sum_{r_*,s_*\in\{0,1\}} (-)^{a_*s_*+b_*r_*+na_*b_*}\hat{\lambda}_* = \mu_* + \mathcal{O}(\hbar) \in \mathbb{C}[[\hbar]]\,, \tag{A.52}$$

labelled by $a_*, b_* \in \{0,1\}$ allows us to write

$$L(z) = L(z;\hat{\mu}_*,\hat{\Lambda}) = \sum_{a_*,b_*\in\{0,1\}} \hat{\mu}_* \sum_{i,j\in\mathbb{Z}_n} L_*^{i,j}(z;\hat{\Lambda}) T_{i,j} \tag{A.53}$$

for

$$L_*^{i,j}(z;\hat{\Lambda}) \\ = \epsilon^{-2^{-1}(ij-a_*i-b_*j)} e^{2\pi i b_* \hat{\Lambda}/n} \frac{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](-2\hat{\Lambda}/n|\tau)\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](z-z_*-2\hat{\Lambda}/n|\tau)}{2\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](-2\hat{\Lambda}/n|\tau)\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z-z_*|\tau)} \quad \text{(A.54)}$$

and $z_* = (a_* + b_*\tau)/2$. Here the subscript $*$ on z_* & $L_*^{i,j}(z,\hat{\Lambda})$ indicates dependence on a_*, b_*. To see this compare the quasi-periodicities and pole structure of the above with equations (A.48) & (A.49). When expressed in terms of the μ_* the constraint (A.44) on the λ_* becomes

$$\sum_{a_*,b_*\in\{0,1\}} \mu_* = 2\,. \quad \text{(A.55)}$$

In this form it's straightforward to take the semi-classical limit. We find that for $(i,j) \not\equiv (0,0)$

$$L_*^{i,j}(z;\hat{\Lambda}) \\ = -\frac{\hbar\rho}{n}\epsilon^{-2^{-1}(ij-a_*i-b_*j)} \frac{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]'(0|\tau)\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](z-z_*|\tau)}{\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](0|\tau)\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z-z_*|\tau)} + \mathcal{O}(\hbar^2)\,. \quad \text{(A.56)}$$

so that, up to a term proportional to the identity at order $\hbar$,

$$L(z) \sim \mathbb{1}_V - \frac{\hbar}{n}\sum_{a_*,b_*\in\{0,1\}} \rho\mu_* \sum_{i,j\in\mathcal{I}_n} \epsilon^{-2^{-1}(ij-a_*i-b_*j)} w_{i,j}(z-z_*)T_{i,j} + \mathcal{O}(\hbar^2)\,. \quad \text{(A.57)}$$

(Recall that $\mathcal{I}_n = \mathbb{Z}_n^2 \setminus \{(0,0)\}$.) Here $w_{i,j}(z)$ is the unique meromorphic function with a single simple pole at the origin obeying

$$\operatorname{Res}_{z=0} w_{i,j}(z) = 1\,, \qquad w_{i,j}(z+1) = \epsilon^i w_{i,j}(z)\,, \qquad w_{i,j}(z+\tau) = \epsilon^j w_{i,j}(z)\,. \quad \text{(A.58)}$$

In terms of θ-functions

$$w_{i,j}(z) = \frac{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]'(0|\tau)\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](z|\tau)}{\theta\left[\begin{smallmatrix}1/2+i/n\\1/2-j/n\end{smallmatrix}\right](0|\tau)\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z|\tau)} \quad \text{(A.59)}$$

Note that ρ essentially restores the scale of the μ_*.

The order $\hbar$ contribution to $L(z)$ exactly matches the classical ℓ-matrix derived in subsection 3.3.1 and appearing in equation (3.80) for $\zeta = 1$ & $\xi = \eta = 0$.

Now we repeat the computation for n even. Let $n = 2m$. The residues at the poles of $\widetilde{L}_*^{i,j}(z;\hat\Lambda)$ are listed below

$$\begin{aligned}&\mathrm{Res}_{z=(a_*+b_*\tau)/2}\,\widetilde{L}_*^{i,j}(z;\hat\Lambda)\\ &\quad= -\frac{1}{n}\epsilon^{-((i-b_*)/2)(j-a_*)}(-)^{a_*s_*+b_*r_*}\delta_{i\equiv b_*\,(2)}\delta_{j\equiv a_*\,(2)}e^{\pi i b_*(2\hat\Lambda+a_*)/n}\Delta(\hat\Lambda|n\tau)\end{aligned} \tag{A.60}$$

for $a_*, b_* \in \{0,1\}$, with Δ as defined in equation (A.50). Rescaling by (A.51) and defining $\hat\mu_*$ as in (A.52), we deduce that

$$L(z) = L(z;\hat\mu,\hat\Lambda) = \sum_{a_*,b_*\in\{0,1\}}\hat\mu_*\sum_{k,l\in\mathbb{Z}_m}L_*^{2k+a_*,2l+b_*}(z)T_{2k+a_*,2l+b_*} \tag{A.61}$$

for

$$\begin{aligned}&L_*^{2k+a_*,2l+b_*}(z;\hat\Lambda)\\ &\quad= \hat\mu_*\epsilon^{-2kl}e^{\pi i b_*(2\hat\Lambda+a_*)/n}\frac{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](-2\hat\Lambda/n|\tau)\,\theta\left[\begin{smallmatrix}1/2+2k/n+a_*/n\\1/2-2l/n-b_*/n\end{smallmatrix}\right](z-z_*-2\hat\Lambda/n|\tau)}{\theta\left[\begin{smallmatrix}1/2+2k/n+a_*/n\\1/2-2l/n-b_*/n\end{smallmatrix}\right](-2\hat\Lambda/n|\tau)\,\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z-z_*|\tau)}\,,\end{aligned} \tag{A.62}$$

Our constraint on the λ_* translates into the condition that $\mu_*|_{a_*=b_*=0} = 1$. In this form we can directly take the classical limit to find that, up to the addition of a term at order $\hbar$ which is proportional to the identity,

$$L(z) \sim \mathbb{1}_V - \frac{2\hbar}{n}\sum_{a_*,b_*\in\{0,1\}}\rho\mu_*e^{\pi i a_*b_*/n}\sum_{k,l\in\mathbb{Z}_m}{}^{'}\,\epsilon^{-2kl}w_{2k+a_*,2l+b_*}(z-z_*)T_{2k+a_*,2l+b_*} + \mathcal{O}(\hbar^2)\,. \tag{A.63}$$

The primed sum indicates that we remove the term involving $T_{0,0}$. ρ is effectively playing the role of $\mu_*|_{a_*=b_*=0}$ in the above.

The order $\hbar$ contribution to $L(z)$ exactly matches the classical ℓ-matrix derived in subsection 3.3.1 and appearing in equation (3.87) for $\zeta = 1$ & $\xi = \eta = 0$.

A.6 Algebra for trigonometric actions

In this appendix we show that the equations of motion derived from the trigonometric action (4.84) are equivalent to the ASDYM equations for the gauge group G, and also verify that it coincides with the action (4.90) of Mason & Sparling if we set $h_+ = h_- = h$.

The equations of motion of the trigonometric action are

$$\Big[\omega_{\alpha_-,\alpha_+}\wedge\partial(\sigma_-^{-1}\tilde{\partial}\sigma_+\sigma_+^{-1}\sigma_-)\Big]_{\mathfrak{g}_+}=0\,,\qquad\Big[\omega_{\alpha_-,\alpha_+}\wedge\tilde{\partial}(\sigma_+^{-1}\partial\sigma_-\sigma_-^{-1}\sigma_+)\Big]_{\mathfrak{g}_-}=0 \quad \text{(A.64)}$$

as given in (4.87). We begin by showing that the ASDYM equations for G imply the above.

Recall Yang's equation for a G-valued field σ

$$\omega_{\alpha_-,\alpha_+}\wedge\partial(\tilde{\partial}\sigma\sigma^{-1})=0 \iff \omega_{\alpha_-,\alpha_+}\wedge\tilde{\partial}(\sigma^{-1}\partial\sigma)=0\,. \quad \text{(A.65)}$$

Using our decomposition $U\times\widetilde{H}=G_-G_+$ we can write $\sigma=\sigma_-^{-1}\sigma_+$ for σ_-,σ_+ taking values in $G_\pm$ respectively. Substituting this into the two forms of Yang's equation we get

$$\omega_{\alpha_-,\alpha_+}\wedge\partial(\sigma_-^{-1}\tilde{\partial}\sigma_+\sigma_+^{-1}\sigma_--\sigma_-^{-1}\tilde{\partial}\sigma_-)=0\,,\qquad\omega_{\alpha_-,\alpha_+}\wedge\tilde{\partial}(\sigma_+^{-1}\partial\sigma_+-\sigma_+^{-1}\partial\sigma_-\sigma_-^{-1}\sigma_+)=0\,. \quad \text{(A.66)}$$

Since $\sigma_\pm^{-1}\tilde{\partial}\sigma_\pm$ takes values in $\mathfrak{g}_\pm$, projecting the above equations onto $\mathfrak{g}_+$ and $\mathfrak{g}_-$ respectively gives

$$\Big[\omega_{\alpha_-,\alpha_+}\wedge\partial(\sigma_-^{-1}\tilde{\partial}\sigma_+\sigma_+^{-1}\sigma_-)\Big]_{\mathfrak{g}_+}=0\,,\qquad\Big[\omega_{\alpha_-,\alpha_+}\wedge\tilde{\partial}(\sigma_+^{-1}\partial\sigma_-\sigma_-^{-1}\sigma_+)\Big]_{\mathfrak{g}_-}=0\,. \quad \text{(A.67)}$$

These are precisely the classical equations of motion (A.64).

Showing the converse is marginally less straightforward. We begin by removing the projections in our equations of motion by writing

$$\omega_{\alpha_-,\alpha_+}\wedge(\partial(\sigma_-^{-1}\tilde{\partial}\sigma_+\sigma_+^{-1}\sigma_-)+\rho_-)=0\,,\qquad\omega_{\alpha_-,\alpha_+}\wedge(\tilde{\partial}(\sigma_+^{-1}\partial\sigma_-\sigma_-^{-1}\sigma_+)+\rho_+)=0\,, \quad \text{(A.68)}$$

for $\rho_\pm\in\mathfrak{g}_\pm$. Conjugating these formulae by σ_- and σ_+ respectively gives

$$\begin{aligned}&\omega_{\alpha_-,\alpha_+}\wedge([\tilde{\partial}\sigma_+\sigma_+^{-1},\partial\sigma_-\sigma_-^{-1}]+\partial(\tilde{\partial}\sigma_+\sigma_+^{-1})+\sigma_-\rho_-\sigma_-^{-1})=0\,,\\&\omega_{\alpha_-,\alpha_+}\wedge([\partial\sigma_-\sigma_-^{-1},\tilde{\partial}\sigma_+\sigma_+^{-1}]+\tilde{\partial}(\partial\sigma_-\sigma_-^{-1})+\sigma_+\rho_+\sigma_+^{-1})=0\,.\end{aligned} \quad \text{(A.69)}$$

Adding these together

$$\omega_{\alpha_-,\alpha_+}\wedge(\partial(\tilde{\partial}\sigma_+\sigma_+^{-1})+\sigma_-\rho_-\sigma_-^{-1}+\tilde{\partial}(\partial\sigma_-\sigma_-^{-1})+\sigma_+\rho_+\sigma_+^{-1})=0 \quad \text{(A.70)}$$

and projecting onto $\mathfrak{g}_{\mp}$ we learn that

$$\omega_{\alpha_-,\alpha_+} \wedge (\tilde{\partial}(\partial\sigma_-\sigma_-^{-1}) + \sigma_-\rho_-\sigma_-^{-1}) = 0\,, \qquad \omega_{\alpha_-,\alpha_+} \wedge (\partial(\tilde{\partial}\sigma_+\sigma_+^{-1}) + \sigma_+\rho_+\sigma_+^{-1}) = 0\,. \tag{A.71}$$

We can clearly solve these equations to get

$$\omega_{\alpha_-,\alpha_+}\wedge\rho_- = -\omega_{\alpha_-,\alpha_+}\wedge\partial(\sigma_-^{-1}\tilde{\partial}\sigma_-)\,, \qquad \omega_{\alpha_-,\alpha_+}\wedge\rho_+ = -\omega_{\alpha_-,\alpha_+}\wedge\tilde{\partial}(\sigma_+^{-1}\partial\sigma_+)\,. \tag{A.72}$$

Substituting these expressions back into (A.68), we recover both forms of Yang's equation.

Next we show that substituting

$$\sigma_-^{-1} = (\ell h, h^{-1})\,, \qquad \sigma_+ = (hu, h)\,, \tag{A.73}$$

into the trigonometric action (4.84)

$$\frac{1}{\langle\alpha_-\ \alpha_+\rangle}\int_{\mathbb{R}^4} \omega_{\alpha_-,\alpha_+} \wedge \mathrm{tr}(\partial\sigma_-\sigma_-^{-1} \wedge \tilde{\partial}\sigma_+\sigma_+^{-1})\,, \tag{A.74}$$

leads to the action (4.90) of Mason & Sparling. We begin by calculating

$$\begin{aligned}
&\mathrm{tr}(\partial\sigma_-\sigma_-^{-1} \wedge \tilde{\partial}\sigma_+\sigma_+^{-1})\\
&= \mathrm{tr}_0(\partial(h^{-1}\ell^{-1})\ell h \wedge \tilde{\partial}(hu)u^{-1}h^{-1}) - \mathrm{tr}_0(\partial hh^{-1} \wedge \tilde{\partial}hh^{-1})\\
&= \mathrm{tr}_0(\partial(h^{-1}\ell^{-1})\ell h \wedge h\tilde{\partial}uu^{-1}h^{-1}) + \mathrm{tr}_0(\partial(h^{-1}\ell^{-1})\ell h \wedge \tilde{\partial}hh^{-1}) - \mathrm{tr}_0(\partial hh^{-1} \wedge \tilde{\partial}hh^{-1})\\
&= \mathrm{tr}_0(\partial(h^{-2}\ell^{-1})\ell h^2 \wedge \tilde{\partial}uu^{-1}) - \mathrm{tr}_0(h^{-1}\partial h \wedge \tilde{\partial}uu^{-1}) - \mathrm{tr}_0(\ell^{-1}\partial\ell \wedge h\tilde{\partial}hh^{-2})\\
&- \mathrm{tr}_0(h^{-1}\partial h \wedge \tilde{\partial}hh^{-1}) - \mathrm{tr}_0(\partial hh^{-1} \wedge \tilde{\partial}hh^{-1})\,.
\end{aligned} \tag{A.75}$$

Noting that $\mathrm{tr}_0(xy) = 0$ for $x \in \mathfrak{h}$ and $y \in \mathfrak{n}_\pm$, and that $\mathfrak{h}$ is abelian, this can be simplified to

$$\mathrm{tr}_0(\partial(h^{-2}\ell^{-1})\ell h^2 \wedge \tilde{\partial}uu^{-1}) - 2\mathrm{tr}_0(\partial hh^{-1} \wedge \tilde{\partial}hh^{-1})\,. \tag{A.76}$$

We can rewrite this, observing that

$$\mathrm{tr}_0(\partial(h^{-2}\ell^{-1})\ell h^2 \wedge \tilde{\partial}(h^{-2}\ell^{-1})\ell h^2) = 4\mathrm{tr}_0(\partial hh^{-1} \wedge \tilde{\partial}hh^{-1})\,, \tag{A.77}$$

as

$$\mathrm{tr}_0(\partial(h^{-2}\ell^{-1})\ell h^2 \wedge \tilde{\partial}uu^{-1}) - \frac{1}{2}\mathrm{tr}_0(\partial(h^{-2}\ell^{-1})\ell h^2 \wedge \tilde{\partial}(h^{-2}\ell^{-1})\ell h^2)\,. \tag{A.78}$$

We therefore obtain

$$\frac{1}{\langle \alpha_- \, \alpha_+ \rangle} \int_{\mathbb{R}^4} \omega_{\alpha_-,\alpha_+} \wedge \operatorname{tr}_0 \left(\partial L L^{-1} \wedge \tilde{\partial} U U^{-1} - \frac{1}{2} \partial L L^{-1} \wedge \tilde{\partial} L L^{-1} \right) \tag{A.79}$$

as required.

Appendix B

Further background

B.1 Review of symmetric pairs

In this appendix we briefly review symmetric pairs, which are fundamental to the construction of the twisted Yangian.

An involutive automorphism σ of the complex simple Lie algebra $\mathfrak{g}$ induces a $\mathbb{Z}_2$ grading, i.e., a decomposition

$$\mathfrak{g} = \mathfrak{h} \dotplus \mathfrak{m} , \tag{B.1}$$

for $\mathfrak{h}$ and $\mathfrak{m}$ the positive and negative eigenspaces of σ, such that

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} , \qquad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m} , \qquad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h} . \tag{B.2}$$

(Recall that $\dotplus$ denotes the direct sum as vector spaces.) The first of these relations tells us that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, while the second tells us $\mathfrak{m}$ admits an action of $\mathfrak{h}$. Conversely, if $\mathfrak{g}$ can be split into a direct sum of $\mathfrak{h}$ and $\mathfrak{m}$ satisfying (B.2), the we can define an involutive automorphism of $\mathfrak{g}$ which acts trivially on $\mathfrak{h}$ and as multiplication by -1 on $\mathfrak{m}$.

We refer to a $\mathbb{Z}_2$ graded Lie algebra as a symmetric pair, and denote it by $(\mathfrak{g}, \mathfrak{h})$.[1] The argument above shows that symmetric pairs are in bijection with involutive automorphisms of $\mathfrak{g}$.

$\mathfrak{h}$ is in general a reductive Lie algebra, i.e., it's of the form $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{t}$ for $\mathfrak{s}$ semisimple and $\mathfrak{t}$ abelian. $\mathfrak{s}$ can then be further decomposed into its simple summands. In practice it turns out that either $\mathfrak{t} = \mathbb{C}$ or it doesn't appear, and that $\mathfrak{s}$ has at most two simple summands.

[1] This is not to be confused with a Lie superalgebra for which the symmetry of the Lie superbracket is tied to the grading.

When working with symmetric pairs it's useful to choose a basis of $\mathfrak{g}$ which is adapted to the decomposition (B.1). Writing $\{t_a\}_{a=1}^{\dim\mathfrak{g}}$ for our basis of $\mathfrak{g}$, we assume that $\{t_\alpha\}_{\alpha=1}^{\dim\mathfrak{h}}$ is a basis of $\mathfrak{h}$ and $\{t_\mu\}_{\mu=\dim\mathfrak{h}+1}^{\dim\mathfrak{g}}$ is a basis of $\mathfrak{m}$. We use Roman letters $a, b, c, \ldots$ to index basis vectors of $\mathfrak{g}$, indices $\alpha, \beta, \gamma, \ldots$ from the beginning of the Greek alphabet to label basis vectors of $\mathfrak{h}$ and indices $\mu, \nu, \xi, \ldots$ from the middle of the Greek alphabet to label basis vectors of $\mathfrak{m}$. We further refine our basis of $\mathfrak{h}$ so that it is adapted to its decomposition into abelian and simple summands. Since $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal, if we raise or lower a Greek index from the start of the alphabet with the bilinear tr it will still belong to the start, and similarly for indices from the middle of the Greek alphabet. Note, however, that the restriction of tr to $\mathfrak{h}$ does not coincide the intrinsic definition of tr on $\mathfrak{h}$.

Using this notation, the conditions in (B.2) can be expressed as

$$f_{\alpha\beta}{}^{\xi} = 0\,, \qquad f_{\alpha\nu}{}^{\gamma} = 0\,, \qquad f_{\mu\nu}{}^{\xi} = 0\,. \tag{B.3}$$

(In fact since we can raise and lower indices with the Killing form the first two conditions are equivalent.)

Since $\mathfrak{g}$ is simple, we have

$$f^{ab}{}_{c} f^{c}{}_{bd} = 2\mathbf{h}^\vee \delta^a{}_d\,. \tag{B.4}$$

(The appearance of the dual Coxeter number on the right hand side is a consequence of our normalization of tr.) From this we can deduce that

$$2\mathbf{h}^\vee \delta^\alpha{}_\delta = f^{\alpha\beta}{}_{\gamma} f^{\gamma}{}_{\beta\delta} + f^{\alpha\nu}{}_{\xi} f^{\xi}{}_{\nu\delta}\,. \tag{B.5}$$

Now, $f^{\alpha\beta}{}_{\gamma} f^{\gamma}{}_{\beta\delta}$ is built from the structure constants of $\mathfrak{h}$ and the restriction of the bilinear tr to $\mathfrak{h}$. Certainly this restriction is an $\mathfrak{h}$-invariant bilinear, and so $f^{\alpha\beta}{}_{\gamma} f^{\gamma}{}_{\beta\delta}$ defines a $\mathfrak{h}$-invariant endomorphism of $\mathfrak{h}$. Schur's lemma then fixes

$$f^{\alpha\beta}{}_{\gamma} f^{\gamma}{}_{\beta\delta} = c_\alpha \delta^\alpha{}_\delta \tag{B.6}$$

for $c_\alpha \in \mathbb{C}$ depending only on which summand of $\mathfrak{h}$ the basis vector t_α belongs to. This then implies that

$$f^{\alpha\nu}{}_{\xi} f^{\xi}{}_{\nu\delta} = \bar{c}_\alpha \delta^\alpha{}_\delta\,, \tag{B.7}$$

where $\bar{c}_\alpha = 2\mathbf{h}^\vee - c_\alpha$. These constants appear in the J-presentation of the twisted Yangian given in [24] and reproduced in equation (2.119).

In this work we'll concentrate on involutive inner automorphisms of the classical simple Lie algebras. Inner automorphisms of $\mathfrak{g}$ are given by conjugation by an element of $G/Z(G)$. We write $\sigma = \operatorname{conj}\tau$ for $\tau \in G$. For σ to be an involution we must have $\tau^2 \in Z(G)$.

All isomorphism classes of symmetric pairs for classical $\mathfrak{g}$ and associated to involutive inner automorphisms are listed in the table below. We have also listed suitable choices of $\tau \in G$ for each symmetric pair. We label them according to Cartan's classification of symmetric spaces [33], and have omitted the trivial cases $\mathfrak{g} = \mathfrak{h}$ for which $\tau = \text{id}.$.

Label	$\mathfrak{g}$	$\mathfrak{h}$	τ
AIII	$\mathfrak{sl}_n(\mathbb{C}) \cong A_{n-1}$	$\mathfrak{sl}_{n-k}(\mathbb{C}) \oplus \mathfrak{sl}_k(\mathbb{C}) \oplus \mathbb{C}$	$e^{i\pi k/n}\text{diag}(\mathbb{1}_{n-k}, -\mathbb{1}_k)$
BI	$\mathfrak{so}_{2n+1}(\mathbb{C}) \cong B_n$	$\mathfrak{so}_{2n+1-k}(\mathbb{C}) \oplus \mathfrak{so}_k(\mathbb{C})$	$(-)^k\text{diag}(\mathbb{1}_{2n+1-k}, -\mathbb{1}_k)$
CI	$\mathfrak{sp}_{2n}(\mathbb{C}) \cong C_n$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$	$\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$
CII		$\mathfrak{sp}_{2(n-k)}(\mathbb{C}) \oplus \mathfrak{sp}_{2k}(\mathbb{C})$	$\text{diag}(\mathbb{1}_{n-k}, -\mathbb{1}_k, \mathbb{1}_{n-k}, -\mathbb{1}_k)$
DI	$\mathfrak{so}_{2n}(\mathbb{C}) \cong D_n$	$\mathfrak{so}_{2(n-k)}(\mathbb{C}) \oplus \mathfrak{so}_{2k}(\mathbb{C})$	$\text{diag}(\mathbb{1}_{2(n-k)}, -\mathbb{1}_{2k})$
DIII		$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$	$\begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$

Table B.1 Inner automorphisms of classical simple Lie algebras

B.2 Matching ℓ- and K-matrices

Unlike R-matrices, K-matrices are rarely studied as formal power series in a parameter $\hbar$. Indeed, we believe that quasi-classical K-matrices and classical ℓ-matrices were first introduced by the author in the papers [29, 30], although similar concepts have implicitly appeared elsewhere. As such, all K-matrices in the literature with which we wish to compare our results involve finite parameters, and we must introduce $\hbar$ by hand. Whilst this is often also necessary for bulk R-matrices, it is more difficult for K-matrices since their classical limits are not necessarily the identity, or even independent of the spectral parameter.

Suppose we are given a K-matrix $K(z, \lambda, \mu)$ which solves the BYBE for an R-matrix $R(z, \lambda)$. Here $\lambda \in \mathbb{C}^M$ are complex parameters appearing in the R-matrix on which the K-matrix may also depend, and $\mu \in \mathbb{C}^N$ are the complex parameters in the K-matrix which are independent of those in the R-matrix.

We must first introduce $\hbar$ dependence in the R-matrix. To do so we follow these steps:

1. Identify appropriate choices of the parameters $\lambda = \lambda_0$ so that $R(z, \lambda_0)$ is proportional to the identity. This is the classical limit.

2. Replace $\lambda \in \mathbb{C}^M$ with $\hat{\lambda} \in \mathbb{C}[[\hbar]]^M$ everywhere in the R-matrix, and take $\hat{\lambda} = \lambda_0 + \mathcal{O}(\hbar)$ so that in the classical limit we recover the identity.

3. Expand $\hat{\lambda} = \lambda_0 + \hbar\lambda_1 + \mathcal{O}(\hbar^2)$ to determine the classical r-matrix

$$R(z, \hat{\lambda}) = \mathbb{1} + \hbar r(z, \lambda_0, \lambda_1) + \mathcal{O}(\hbar^2)\,. \tag{B.8}$$

   It is often convenient to eliminate one of the parameters in λ_1 by identifying it with $\hbar$.

To introduce $\hbar$ dependence in the K-matrix we follow a similar sequence of steps:

1. Identify appropriate choices of the parameters $\mu = \mu_0$ for which $K(z, \lambda_0, \mu_0)$ is proportional to $\tau(z)$ obeying $\tau(z)\tau(-z) \propto \mathbb{1}$. This is the classical limit. Usually τ is constant in z.

2. Replace $\lambda \in \mathbb{C}^M$ and $\mu \in \mathbb{C}^N$ with $\hat{\lambda} \in \mathbb{C}[[\hbar]]^M$ and $\hat{\mu} \in \mathbb{C}[[\hbar]]^N$ everywhere in the K-matrix. Take $\hat{\mu} = \mu_0 + \mathcal{O}(\hbar)$ so that in the classical limit we recover $\tau(z)$.

3. Expand $\hat{\mu} = \mu_0 + \hbar\mu_1 + \mathcal{O}(\hbar^2)$ and similarly for $\hat{\lambda}$ to obtain the classical ℓ-matrix

$$K(z, \hat{\lambda}, \hat{\mu})\tau(z)^{-1} = \mathbb{1} + \hbar\ell(z, \lambda_0, \mu_0, \lambda_1, \mu_1) + \mathcal{O}(\hbar^2)\,. \tag{B.9}$$

We are always free to multiply an R- or K-matrix by some formal power series in $\hbar$ with coefficients in the ring of meromorphic functions on $\mathbb{C}$. Multiplying by $1 + f(z)\hbar + \mathcal{O}(\hbar^2)$ has the effect of shifting the classical r- or ℓ-matrix by a term proportional to $\mathbb{1}$.

B.3 Notation and conventions for spinors

In this appendix we review our notation and conventions for spinors. These are used exclusively in chapters 4 & 5.

All indices will be regarded as 'abstract' in the sense that V^a refers to a particular vector, not its components in some basis.

Roman indices $a, b, c, \ldots$ from the beginning of the alphabet label elements of the tangent (and cotangent) bundles to 4d complexified spacetime, $\mathbb{CM}^4 = \mathbb{C}^4$. Fixing a real structure on $\mathbb{CM}^4$ these become labels for elements of the tangent (and cotangent) bundles to a real form of $\mathbb{CM}^4$, e.g. 4d Euclidean space $\mathbb{R}^4$. They are contracted using the standard flat metric g_{ab}. We also make use of the invariant alternating tensor ε_{abcd} with $\varepsilon_{0123} = 1$. Primed and unprimed capital indices $A', B', C', \ldots$ and $A, B, C, \ldots$ label elements of $\mathbb{S}^{\pm}$, the left- and right-handed spin representations of $\mathrm{SO}_4(\mathbb{C})$ respectively. They can be similarly interpreted as spin representations over a real form of $\mathrm{SO}_4(\mathbb{C})$. They are contracted using the $\mathrm{SL}_2(\mathbb{C})$-invariant tensors $\varepsilon_{A'B'}$ and ε_{AB} where $\varepsilon_{0'1'} = \varepsilon_{01} = 1$. We define

$$\langle \alpha\, \beta \rangle = \alpha^{A'} \beta^{B'} \varepsilon_{A'B'} = \alpha^{A'} \beta_{A'} \quad [\mu\, \nu] = \mu^A \nu^B \varepsilon_{AB} = \mu^A \nu_A \,. \tag{B.10}$$

The isomorphism $\mathbb{CM}^4 \cong \mathbb{S}^+ \otimes \mathbb{S}^-$ allows us to write $V^a = V^{AA'}$. Then

$$g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'} \,, \qquad \varepsilon_{abcd} = \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'} \,. \tag{B.11}$$

We use Roman indices $i, j, k, \ldots$ from the middle of the alphabet to label elements of the tangent (and cotangent) bundles to 3d complexified Minkowski space, $\mathbb{CM}^3$. We can also view these as indices on real forms of $\mathbb{CM}^3$, e.g., 3d Euclidean space $\mathbb{R}^3$. They are contracted using the standard flat metric g_{ij}, and we also make use of the invariant alternating tensor ε_{ijk} with $\varepsilon_{123} = 1$. We abuse notation by using unprimed spinor indices $A', B', C', \ldots$ as labels for elements of $\mathbb{S}$, the spin representation of $\mathrm{SO}_3(\mathbb{C})$. This is justified by viewing $\mathbb{S}^+$ as a representation of $\mathrm{SO}_3(\mathbb{C})$ by embedding it into $\mathrm{SO}_4(\mathbb{C})$. These indices are contracted using $\varepsilon_{A'B'}$ as above. The isomorphism $\mathbb{CM}^3 \cong S^2\mathbb{S}$ allows us to write $V^i = V^{A'B'}$ where the right hand side is symmetric under exchange of A' and B'. Then

$$g_{ij} = \frac{1}{2}(\varepsilon_{A'C'}\varepsilon_{B'D'} + \varepsilon_{A'D'}\varepsilon_{B'C'}) \,, \quad \varepsilon_{ijk} = \frac{1}{\sqrt{2}}(\varepsilon_{A'C'}\varepsilon_{E'B'}\varepsilon_{D'F'} - \varepsilon_{A'F'}\varepsilon_{C'E'}\varepsilon_{B'D'}) \,, \tag{B.12}$$

where we are identifying $i = A'B'$, $j = C'D'$ and $k = E'F'$.

We use Greek indices $\alpha, \beta, \gamma, \ldots$ to label twistor indices.

B.4 Homogeneous coordinates on $\mathbb{CP}^1$

Here we review the use homogeneous coordinates on $\mathbb{CP}^1$. These are used exclusively in chapters 4 & 5.

Let $\pi^{A'} = (\pi^{0'}, \pi^{1'}) \in \mathbb{C}^2 \setminus \{0\}$ be a non-vanishing left-handed spinor. We represent its equivalence class under the relation $\pi^{A'} \sim t\pi^{A'}$ for $t \in \mathbb{C}^*$ by $[\pi] \in \mathbb{CP}^1$. A function F of $\pi^{A'}$ and its complex conjugates is said to have holomorphic weight m

and antiholomorphic weight n if under this rescaling $F \mapsto t^m \bar{t}^n F$. We can interpret F as a smooth section of the line bundle $\mathcal{O}(m) \otimes \bar{\mathcal{O}}(n) \to \mathbb{CP}^1$.

Rescalings of π are generated by the vector field $\Gamma = \pi^{A'} \partial_{\pi^{A'}}$ and its conjugate $\bar{\Gamma}$. We can identify $T^{1,0}_{[\pi]}\mathbb{CP}^1$ with the quotient of $T^{1,0}_{\pi}\mathbb{C}^2$ by the subspace generated by Γ. These subspaces generate a subbundle $\langle \Gamma \rangle \subset T^{1,0}\mathbb{C}^2$. Sections of $T^{1,0}\mathbb{CP}^1 \otimes \mathcal{O}(m) \otimes \bar{\mathcal{O}}(n)$ are then realised as sections of the quotient bundle $T^{1,0}\mathbb{C}^2/\langle \Gamma \rangle$ with holomorphic weight m and antiholomorphic weight n. The line bundle $T^{1,0}\mathbb{CP}^1 \otimes \mathcal{O}(-2)$ has a unique holomorphic section given by

$$\partial_0 = \left[-\frac{\alpha^{A'}}{\langle \pi\, \alpha \rangle} \frac{\partial}{\partial \pi^{A'}} \right], \tag{B.13}$$

for $[\alpha] \in \mathbb{CP}^1$ an arbitrary choice of reference spinor. The above definition makes sense only for $[\pi] \neq [\alpha]$, but the equivalence class on the right hand side is actually independent of α and so can be used to define ∂_0 globally. Of course, this whole discussion goes through identically for the antiholomorphic tangent bundle if we replace Γ by $\bar{\Gamma}$ everywhere.

The holomorphic cotangent space $T^{*\,1,0}_{[\pi]}\mathbb{CP}^1$ can be realised as the kernel of $\Gamma \lrcorner$ in $T^{*\,1,0}_{\pi}\mathbb{C}^2$. This defines a subbundle $\ker(\Gamma \lrcorner) \subset T^{*\,1,0}\mathbb{C}^2$. Sections of $T^{*\,1,0}\mathbb{CP}^1 \otimes \mathcal{O}(m) \otimes \bar{\mathcal{O}}(n)$ can be identified with sections of the subbundle $\ker(\Gamma \lrcorner)$ with holomorphic and antiholomorphic weights m and n respectively. The line bundle $T^{*\,1,0}\mathbb{CP}^1 \otimes \mathcal{O}(2)$ has a unique holomorphic section

$$e^0 = \langle \mathrm{d}\pi\, \pi \rangle\,. \tag{B.14}$$

The antiholomorphic cotangent space and higher degree forms can be incorporated into this description in the obvious way.

Fixing a dyad of left-handed spinors $\{\alpha, \beta\}$ we may introduce inhomogeneous coordinates on $\mathbb{CP}^1$ by

$$\pi^{A'} \sim \alpha^{A'} - \zeta \beta^{A'} \tag{B.15}$$

where $\zeta = \langle \pi\, \alpha \rangle / \langle \pi\, \beta \rangle$. Then

$$\mathrm{d}\zeta = \frac{e^0}{\langle \pi\, \beta \rangle^2}\,, \qquad \frac{\partial}{\partial \zeta} = \langle \pi\, \beta \rangle^2 \partial_0\,. \tag{B.16}$$

Using these identities we can easily transition from homogeneous to inhomogeneous coordinates.

B.5 The ASDYM equations

The ASDYM equations for a connection $\nabla = \mathrm{d} + A$ on a principal G-bundle over complexified spacetime $\mathbb{CM}^4 = \mathbb{C}^4$ are given by

$$* F = -F \,, \tag{B.17}$$

for $*$ the Hodge star operator induced by the flat metric g and F the curvature of ∇. The ASDYM equations imply the Yang-Mills equations as a consequence of the Bianchi identity on F,

$$\nabla * F = -\nabla F = 0 \tag{B.18}$$

Since the curvature is an anti-symmetric tensor it can be decomposed as

$$F_{ab} = \varepsilon_{AB} F_{A'B'} + \varepsilon_{A'B'} F_{AB} \tag{B.19}$$

for F_{AB} and $F_{A'B'}$ symmetric in their indices. The first term in this sum is self-dual (SD), and the second is ASD, and so the ASDYM equations can be expressed as

$$F_{A'B'} = -\frac{1}{2} \varepsilon^{AB} [\nabla_{AA'}, \nabla_{BB'}] = 0 \,. \tag{B.20}$$

Alternatively they are equivalent to the vanishing of

$$\pi^{A'} \pi^{B'} [\nabla_{AA'}, \nabla_{BB'}] = \varepsilon_{AB} \pi^{A'} \pi^{B'} F_{A'B'} \tag{B.21}$$

for all $[\pi] \in \mathbb{CP}^1$. This is the statement that the restriction of ∇ to any SD 2-plane in $\mathbb{CM}^4$ is flat. It's this observation that allowed Ward to relate solutions of the ASDYM equations to holomorphic vector bundles over the twistor space of $\mathbb{CM}^4$ [164].

There are two 2^{nd} order forms for the ASDYM equations frequently discussed in the literature. Both require partially breaking Lorentz invariance.

To obtain the first fix a dyad $\langle \alpha \, \beta \rangle = 1$. We can solve

$$\alpha^{A'} \alpha^{B'} [\nabla_{AA'}, \nabla_{BB'}] = 0 \,, \tag{B.22}$$

by writing $\alpha^{A'} A_{AA'} = -\alpha^{A'} \partial_{AA'} \sigma_\alpha \sigma_\alpha^{-1}$ for some $\sigma_\alpha : \mathbb{CM}^4 \to G$. Similarly

$$\beta^{A'} \beta^{B'} [\nabla_{AA'}, \nabla_{BB'}] = 0 \,, \tag{B.23}$$

is solved by $\beta^{A'} A_{AA'} = -\beta^{A'} \partial_{AA'} \sigma_\beta \sigma_\beta^{-1}$ for some $\sigma_\beta : \mathbb{CM}^4 \to G$. A gauge transformation by σ_β^{-1} fixes $\beta^{A'} A_{AA'} = 0$, and we find that

$$A_{AA'} = -\beta_{A'} \alpha^{B'} \partial_{AB'} \sigma \sigma^{-1} \tag{B.24}$$

for $\sigma = \sigma_\beta^{-1} \sigma_\alpha$. The ASDYM equations for A then follow from

$$\varepsilon^{AB} \alpha^{A'} \beta^{B'} [\nabla_{AA'}, \nabla_{BB'}] = \varepsilon^{AB} \alpha^{A'} \beta^{B'} \partial_{BB'} (\partial_{AA'} \sigma \sigma^{-1}) = 0\,. \tag{B.25}$$

This is Yang's equation, and σ is referred to as Yang's matrix [176].

For the second we choose a left-handed spinor α, which we use to specify the gauge $\alpha^{A'} A_{AA'} = 0$. This is solved by $A_{AA'} = \xi_A \alpha_{A'}$ for some right-handed spinor field ξ. Next we impose

$$\varepsilon^{AB} \alpha^{A'} [\nabla_{AA'}, \nabla_{BB'}] = \alpha_{B'} \varepsilon^{AB} \alpha^{A'} \partial_{AA'} \xi_B = 0 \implies \alpha^{A'} \partial_{AA'} \xi^A = 0\,. \tag{B.26}$$

From the Poincaré Lemma we can then write $\xi_A = \alpha^{A'} \partial_{AA'} \phi$ for ϕ a $\mathfrak{g}$-valued scalar field. The ASDYM equations follow from

$$\Delta \phi = \varepsilon^{AB} \alpha^{A'} \alpha^{B'} [\partial_{AA'} \phi, \partial_{BB'} \phi]\,, \tag{B.27}$$

for $\Delta = g^{ab} \partial_a \partial_b$ the Laplacian. Equivalently

$$\mathrm{d} * \mathrm{d}\phi + \omega_{\alpha,\alpha} \wedge \mathrm{d}\phi \wedge \mathrm{d}\phi = 0\,, \tag{B.28}$$

for $\omega_{\alpha,\alpha} = \mathrm{d}^2 x^{A'B'} \alpha_{A'} \alpha_{B'}$.

B.6 The twistor correspondence in indefinite signature

The twistor space of complexified spacetime, $\mathbb{CM}^4 = \mathbb{C}^4$, is $\mathbb{PT} = \mathbb{CP}^3 \setminus \mathbb{CP}^1$. Each point $Z = (\omega, \pi) \in \mathbb{PT}$ defines a totally null 2-plane in $\mathbb{CM}^4$ with self-dual tangent bivector by

$$\omega^A = x^{AA'} \pi_{A'}\,. \tag{B.29}$$

Such 2-planes are referred to as α-planes, and their tangent bivectors are proportional to $\varepsilon^{AB} \pi^{A'} \pi^{B'}$. Conversely, fixing $x \in \mathbb{CM}^4$ in the incidence relation (B.29) and letting $Z = (\omega, \pi)$ vary defines the holomorphic line $\mathbb{CP}_x \xrightarrow{\iota_x} \mathbb{PT}$. We therefore have a double fibration of a correspondence space: the set of pairs $(x, Z) \in \mathbb{CM}^4 \times \mathbb{PT}$ such that equation (B.29) holds.

We can view this correspondence space as an open subset of the flag manifold $\mathbb{F}_{(1,2)}(\mathbb{C}^4)$, which is the set of vector subspaces $E_1 \subset E_2 \subset \mathbb{C}^4$ such that $\dim E_1 = 1$ and $\dim E_2 = 2$. We may interpret E_1 as a point $Z \in \mathbb{CP}^3$, and E_2 as a subspace $\mathbb{CP}^1 \subset \mathbb{CP}^3$ containing E_1. Under the additional assumption that both E_1 and E_2 are contained in $\mathbb{PT} \subset \mathbb{CP}^3$ we can identify E_1 with a point in $\mathbb{PT}$, and E_2 with $\mathbb{CP}_x$ for a unique $x \in \mathbb{CM}^4$. The condition $E_1 \subset E_2$ is equivalent to the incidence relation (B.29). We therefore identify the open subset of $\mathbb{F}_{(1,2)}(\mathbb{C}^4)$ defined by $E_1, E_2 \subset \mathbb{PT}$ with the correspondence space, and denote it by $\mathbb{F}_{(1,2)}\mathbb{T}$. It is double fibred over $\mathbb{PT}$ and $\mathbb{CM}^4$.

$$\begin{array}{ccc} & \mathbb{F}_{(1,2)}\mathbb{T} & \\ \swarrow & & \searrow \\ \mathbb{PT} & & \mathbb{CM}^4 \end{array} \tag{B.30}$$

This is the twistor correspondence for $\mathbb{CM}^4$.

Another interpretation of the correspondence space is as the left-handed projective spin bundle over complexified spacetime, $\mathbb{PS}^+ \to \mathbb{CM}^4$. We can see this by using the coordinates (x, π) on $\mathbb{F}_{(1,2)}\mathbb{T}$ with ω determined by the incidence relation (B.29).

Given a subspace $\mathcal{M} \subset \mathbb{CM}^4$ we can define a corresponding twistor space $\mathcal{PT} = \{Z \in \mathbb{PT} | Z \cap \mathcal{M} \neq \emptyset\}$, and have an associated twistor correspondence.

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\rho}\swarrow & & \searrow^{\pi} \\ \mathcal{PT} & & \mathcal{M} \end{array} \tag{B.31}$$

Here $\mathcal{F}$ is the set of pairs $(x, Z) \in \mathcal{M} \times \mathcal{PT}$ obeying the incidence relation (B.29). $\mathcal{F}$ can naturally be viewed as a subset of $\mathbb{F}_{(1,2)}\mathbb{T}$.

Euclidean, Minkowski and ultrahyperbolic 4 dimensional spacetimes can be realised as real forms of $\mathbb{CM}^4$, i.e., as the fixed points of an antiholomorphic involution $\mathbb{CM}^4 \to \mathbb{CM}^4$. On a real slice the complexified Lorentz group, $\mathrm{SO}_4(\mathbb{C})$, reduces to a real form, as does the spin group $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. We can encode a choice of real slice in a complex conjugation on spinors, which then extends to act on twistors and spacetime coordinates by antilinearity.

- Euclidean space $\mathbb{R}^4$: Spinor conjugation is given by

$$\omega^A \mapsto \hat{\omega}^A = (-\overline{\omega^1}, \overline{\omega^0}) \ , \qquad \pi^{A'} \mapsto \hat{\pi}^{A'} = (-\overline{\pi^{1'}}, \overline{\pi^{0'}}) \, . \tag{B.32}$$

  It preserves left-handed and right-handed spinors, and has no fixed points. The spin group is broken to $\mathrm{SU}_2 \times \mathrm{SU}_2$. Conjugation on twistors is

$$Z^\alpha \mapsto \hat{Z}^\alpha = (\omega^A, \pi_{A'}) \, , \tag{B.33}$$

  which also has no fixed points.

- Minkowski space $\mathbb{R}^{1,3}$: Spinor complex conjugation is given by

$$\omega^A \mapsto \bar{\omega}^{A'} = (\overline{\omega^0}, \overline{\omega^1}) \ , \qquad \pi^{A'} \mapsto \bar{\pi}^{A} = (\overline{\pi^{0'}}, \overline{\pi^{1'}}) \, . \tag{B.34}$$

  It swaps left-handed and right-handed spinors. The spin group is broken to a single $\mathrm{SL}_2(\mathbb{C})$. The extension to twistors exchanges twistor space and its dual

$$Z^\alpha \mapsto \bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'}) \, . \tag{B.35}$$

- Ultrahyperbolic space $\mathbb{R}^{2,2}$: Spinor conjugation is given by

$$\omega^A \mapsto \bar{\omega}^A = (\overline{\omega^0}, \overline{\omega^1}) \ , \qquad \pi^{A'} \mapsto \bar{\pi}^{A'} = (\overline{\pi^{1'}}, \overline{\pi^{0'}}) \, . \tag{B.36}$$

  It preserves left-handed and right-handed spinors, and fixes a $\mathbb{RP}^1$ subspace of each. The spin group is broken to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Conjugation on twistors is

$$Z^\alpha \mapsto \bar{Z}^\alpha = (\bar{\omega}^A, \bar{\pi}_{A'}) \, , \tag{B.37}$$

  which fixes the intersection $\mathbb{PT} \cap \mathbb{RP}^3$.

As subspaces of $\mathbb{CM}^4$ we can construct twistor spaces and correspondences for each of the 3 real forms.

- Euclidean space $\mathbb{R}^4$: As was discussed in section 4.1, the twistor correspondence simplifies dramatically in Euclidean signature. We find that $\mathcal{PT} = \mathbb{PT}$, and the α-plane defined by $Z \in \mathbb{PT}$ contains a unique $x \in \mathbb{R}^4$. This x is characterised by the fact that $\mathbb{CP}^1_x$ is the projective line connecting Z and $\hat{Z}$. $\mathbb{PT}$ coincides with $\mathcal{F}$, the left-handed projective spin bundle over $\mathbb{R}^4$.

- Minkowski space $\mathbb{R}^{1,3}$: $\mathcal{PT} = \mathbb{PN} = \{(\omega^A, \pi_{A'}) \in \mathbb{PT} \,|\, \omega^A \bar{\pi}_A - \pi_{A'} \bar{\omega}^{A'} = 0\}$. A point in $\mathbb{PN}$ defines a light ray in $\mathbb{R}^{1,3}$. Again we can identify $\mathcal{F}$ with the left-handed projective spin bundle over $\mathbb{R}^{1,3}$.

- Ultrahyperbolic space $\mathbb{R}^{2,2}$: $\mathcal{PT} = \mathbb{PT}$. The fixed points under conjugation define a real subspace $\mathcal{PT}_{\mathbb{R}} = \mathbb{PT} \cap \mathbb{RP}^3$. Points in $\mathcal{PT}_{\mathbb{R}}$ correspond to α-planes which lie wholly within $\mathbb{R}^{2,2}$, and with real tangent bivectors $\varepsilon^{AB}\pi^{A'}\pi^{B'}$ for $\pi = \bar{\pi}$. The complement of $\mathcal{PT}_{\mathbb{R}}$ in $\mathcal{PT}$ fibres over $\mathbb{R}^{2,2}$. In particular, given $Z \in \mathcal{PT} \setminus \mathcal{PT}_{\mathbb{R}}$ the projective line connecting Z and $\bar{Z}$ is $\mathbb{CP}^1_x$ for some $x \in \mathbb{R}^{2,2}$. As in the previous two cases we can identify $\mathcal{F}$ with the left-handed projective spin bundle over $\mathbb{R}^{2,2}$. There is, however, a distinguished real subspace $\mathcal{F}_{\mathbb{R}} \subset \mathcal{F}$ fixed by spinor conjugation which is an S^1 bundle over $\mathbb{R}^{2,2}$.

For further details see, e.g., [90, 124].